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THE

MESSENGER OF MATHEMATICS.

EDITED BY

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VOL. XL.

[May 1910—April 1911].

12205-1
1 5 12

Cambridge: BOWES & BOWES.
London: MACMILLAN & CO. LTD.
Glasgow: JAMES MACLEHOSE & SONS.

1911.

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MESSENGER OF MATHEMATICS.

HIGH SEXTAN FACTORISATIONS.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's College, London.

(Continued from Vol. xxxix., page 128.)

[The author is indebted to Mr H. J. Woodall, A.R.C.Sc., for help in reading the Proof-sheets, and for useful suggestions.]

Chapter IV. *Diophantine Factorisation.*

53. *Diophantine Factorisation.* This Chapter contains the development of a general Method* of factorisation of Sextans depending on the reduction of N_{vi} to one of the 2^{ic} forms—

$$N_{vi} = P^2 - Q^2 = a^2 + b^2 = A^2 + 3B^2 = A'^2 - 3B'^2 \ldots\ldots\ldots\ldots (166),$$

—where the (P, Q), (a, b), (A, B), (A', B') are *arithmetic* partitions different from the similar known (algebraic) partitions (Art. 3) of N_{vi}—by the solution of certain 2^{ic} Diophantine Equations of type

$$Rx^2 + Sy^2 = z^2 \ldots\ldots\ldots\ldots (167).$$

Each such Diophantine—if solvable at all—will yield an infinity of solutions (x, y, z), and each such solution yields a factorisable Sextan. This evidently opens up a wide field for factorisation of large Sextans.

54. *Dimorph 2^{ic} Partitions.* The *algebraic* 2^{ic} partitions of N_{vi} will be written as before (Art. 3) in Roman letters, and the *arithmetic* partitions isomorph therewith will be written with the similar italic letters, thus

$$N_{vi} = \mathrm{a}^2 + \mathrm{b}^2 = \mathrm{A}^2 + 3\mathrm{B}^2 = \mathrm{A}'^2 - 3\mathrm{B}'^2, \quad \text{[algebraical]} \ldots\ldots (168a),$$
$$= a^2 + b^2 = A^2 + 3B^2 = A'^2 - 3B'^2, \quad \text{[arithmetical]} \ldots\ldots (168b).$$

The second set can of course exist only when N is composite (Art. 6), and any one of the dimorph pairs will suffice for factorisation of N_{vi} by known Rules.

* The Method is one of great generality, applicable to many kinds of 4^{ic} functions (*e.g.* 4-tans, 5-tans, 6-tans, &c.): its application to 4-tans $(x^4 + y^4)$ has already been given in the author's Papers on *Quartan Factorisation*, &c., in Vols xxxvi. and xxxviii. of this Journal.

55. *Diophantine Factors.* The co-factors of N_{vi} resulting from this process will be styled *Diophantine Factors*, and will be denoted by L, M,* so that

$$N_{vi} = P^2 - Q^2 = L.M\,; \quad L = P - Q,\ M = P + Q \ldots\ldots\ldots\ldots (169).$$

They are of course expressible in the (a, b), (A, B), (A', B'), 2^{ic} forms, (being factors of N_{vi}, see Art. 9*a*) ; but as a rule only arithmetically, (not algebraically).

[They are analogous to the *Aurifeuillan Factors* of Art. 35—52, but differ therefrom in not being necessarily prime to one another].

Use of Colon (:). When written close together (in arithmetical work), they will be separated by a colon (:), which is then to be read as a *multiplication-symbol*, thus $N = L : M.$, (*Cf.* Art. 35).

56. *Factorisation-Rules.* When any of the (arithmetic) 2^{ic} partitions (168*b*) have been formed, then N_{vi} can be written—[by aid of (168*a*)]—in one of the forms

$$N_{vi} = P^2 - Q^2 = \frac{(ab)^2 - (a\mathfrak{b})^2}{b^2 - \mathfrak{b}^2} = \frac{(A\,B)^2 - (A\mathfrak{B})^2}{B^2 - \mathfrak{B}^2} = \frac{(A'B')^2 - (A'\mathfrak{B}')^2}{B'^2 - \mathfrak{B}'^2} \quad \ldots\ldots (170),$$

where the denominators can be changed respectively into

$$(a^2 - a^2),\ \tfrac{1}{3}(A^2 - A^2),\ \tfrac{1}{3}(A'^2 - A'^2) \ldots\ldots\ldots\ldots\ldots\ldots (170a),$$

and the factorisation of N_{vi} (into two cofactors) is here obvious; for each numerator, as well as each denominator is here a difference of squares.

56*a*. 2^{ic} *partitions of* L, M. The formulæ (170), though simple and direct (for factorisation purposes) have the practical inconvenience of involving the products of ab, &c., and thus require the computation of numbers $> N_{vi}$ itself. The following formulæ, which give the 2^{ic} partitions of L, M, are free from this defect. Let

$$L = \alpha_1'^2 + \beta_1'^2 = \eta_1'^2 + 3\theta_1'^2 = \chi_1'^2 - 3\psi_1'^2 \ldots\ldots\ldots\ldots (171a),$$

$$M = \alpha_2''^2 + \beta_2''^2 = \eta_2''^2 + 3\theta_2''^2 = \chi_2''^2 - 3\psi_2''^2 \ldots\ldots\ldots\ldots (171b).$$

Then the *ratios* of two "2^{ic} parts" of each partition of L, M are given by either of the values below

$$\frac{\alpha_1'}{\beta_1'} = \frac{a + \mathfrak{a}}{\mathfrak{b} - b} = \frac{\mathfrak{b} + b}{a - \mathfrak{a}}, \quad \frac{\alpha_2''}{\beta_2''} = \frac{a + \mathfrak{a}}{\mathfrak{b} + b} = \frac{\mathfrak{b} - b}{a - \mathfrak{a}} \ldots\ldots\ldots\ldots (172),$$

$$\frac{\eta_1'}{\theta_1'} = \frac{A + \mathfrak{A}}{\mathfrak{B} - B} = 3\frac{\mathfrak{B} + B}{\mathfrak{A} - A}, \quad \frac{\eta_2''}{\theta_2''} = \frac{A + \mathfrak{A}}{\mathfrak{B} + B} = 3\frac{\mathfrak{B} - B}{\mathfrak{A} - A} \ldots\ldots\ldots\ldots (173),$$

$$\frac{\chi_1'}{\psi_1'} = \frac{A' + \mathfrak{A}'}{B' - \mathfrak{B}'} = 3\frac{B' + \mathfrak{B}'}{A' - \mathfrak{A}'}, \quad \frac{\chi_2''}{\psi_2''} = \frac{A' + \mathfrak{A}'}{B' + \mathfrak{B}'} = 3\frac{B' - \mathfrak{B}'}{A' - \mathfrak{A}'} \ldots\ldots\ldots\ldots (174),$$

* This L and M must not be confused with the L and M used in the 2^{ic} partitions (5*h*, &c.): these two usages will not occur together in what follows.

and, since the "2^{ic} parts" of each partition are always mutually prime, it follows that when the above fractions have been *reduced to their lowest terms*, then

The reduced numerators give α_1', α_2''; η_1', η_2''; χ_1', χ_2''.....(175*a*),

The reduced denominators give β_1', β_2''; θ_1', θ_2''; ψ_1', ψ_2'' .. ..(175*b*).

The formulæ (172–174) are liable to variation owing to the double signs (±) of the "2^{ic} parts" in the formulæ (168*a*, *b*) for N:

[As the signs are independent, this gives four variations for each of the formulæ; this is liable to cause some trouble, as the correct sign can (so far as is yet known) only be found by trial, *i.e.* by testing the results in the equation $N = L.M$].

56*b*. *Use of* (A^2-3B^2) *formulæ*. An adjustment is sometimes required in using the formulæ (174) involving (A', B'), (*A'*, *B'*).

As N is supposed to be *odd*, and L, M therefore also *both odd*, it follows that

(γ_1', δ_1') should be one *odd*, one *even*; (γ_2', δ_2') should be one *odd*, one *even*.(176).

Hence, also,

A', *A'* should be both *odd*, or both *even*.....176*a*),

B', *B'* should be both *even*, or both *odd*.....(176*b*).

Of these, one pair (say A', B') is supposed *given* (algebraically): as to the other pair (*A'*, *B'*), its automorphs being given by—(Art. 6).

$$A'^2-3B'^2 = A_1'^2-3B_1'^2 = A_2'^2-3B_2'^2 = \&c.....(177),$$

(obtained by multiplying by the unit-form $2^2-3\,1^2=+1$),

The *A'* "parts" will alternate *odd* and *even*, and..... ..(177*a*),

The *B'* "parts" will alternate *even* and *odd* .....(177*b*).

Any automorph may be chosen* which satisfies the condition (176); the lowest being usually the most convenient for practical calculations.

57. *Diophantine Partitions*. Let the Sextan (N_{vi}) be written in the form

$$N = (x^2+Cy^2)^2+\mu y^2.(R.x^2+Sy^2) \quad .....(178).$$

This will be reducible to one of the four 2^{ic} forms

i. (P^2-Q^2); ii. (a^2+b^2); iii. (A^2+3B^2); iv. $(A'^2-3B'^2)$. .(179),

in the following ways i., ii., iii., iv., respectively—

i. $\mu=-1$; $N=(x^2+Cy^2)^2-y^2.\{(2C'+1)x^2+(C'^2-1)y^2\}$

$$= P^2-Q^2 \quad .....(180),$$

if $\quad (2C'+1)x^2+(C'^2-1)y^2=z^2$ .....(180*a*).

* See *Ex.* iv.—(8), Tab XIII*a*, Art 76 (page 23).

ii. $\mu=+1$; $N=(x^2+C''y^2)^2+y^2.\{-(2C''+1)x^2-(C''^2-1)y^2\}$
$= a^2+b^2$ .....(181),
if $-(2C''+1)x^2-(C''^2-1)y^2=z''^2$.....(181*a*).

iii. $\mu=+3$; $N=(x^2+C'''y^2)+3y^2.\{-\frac{1}{3}(2C'''+1)x^2-\frac{1}{3}(C'''^2-1)y^2\}$
$= A^2+3B^2$.....(182),
if $-\frac{1}{3}(2C'''+1)x^2-\frac{1}{3}(C'''^2-1)y^2=z''^2$.....(182*a*).

iv. $\mu=-3$; $N=(x^2+C^{iv}y^2)^2-3y^2.\{\frac{1}{3}(2C^{iv}+1)x^2+\frac{1}{3}(C^{iv2}-1)y^2\}$
$= A'^2-3B'^2$.....(183),
if $\frac{1}{3}(2C^{iv}+1)x^2+\frac{1}{3}(C^{iv2}-1)y^2=z^{iv2}$.....(183*a*).

58. *Factorisant, Characteristic.* The four Diophantine equations (180*a*)—(183*a*) will be styled *Factorisants* from the property that any solution (x, y, z) thereof yields usually an (arithmetic) 2^{ic} partition of one of the forms i., ii., iii., iv., and therefore leads—(along with the known similar algebraic partition)—to a factorisable Sextan (N_{vi}) and yields at the same the two co-factors (L, M) thereof.

The quantities C', C'', C''', C^{iv} which characterise the several Factorisants will be styled their *Characteristics.* The characteristics and their corresponding factorisants will be said to be of 1st, 2nd, 3rd, and 4th *Class* respectively, according as they fall under Cases i., ii., iii., iv. above.

The four factorisants may be included under the single type

$$Rx^2+Sy^2=z^2 \quad \text{.....(184),}$$

where R, S are functions of the characteristic (C).

59. 2^{ic} *parts.* When a solution (x, y, z) of any Factorisant has been obtained, the 2^{ic} parts

$$(P, Q),\ (a, b),\ (A, B),\ (A', B'),$$

of the (arithmetic) partition required for factorisation are given directly by

i. $P=\pm(x^2+C'y^2)$, $Q=yz'$ .....(185*a*),

ii. a or $b=\pm(x^2+C''y^2)$, b or $a=yz''$ .....(185*b*),

iii. $A=\pm(x^2+C'''y^2)$, $B=yz'''$ .....(185*c*),

iv. $A'=\pm(x^2+C^{iv}y^2)$, $B'=yz^{iv}$ .....(185*d*).

60. *Fractional Symbols.* To give this Method the full advantage of its generality, it is necessary now to remove the

restriction (of Art. 1*a*, *b*) that all symbols should be integers: so that, in what follows—

C, R, S, Z; P, Q; a, b; A, B; A', B', &c. may be fractions...(186),

but x, y, Cy^2, yz should be integers*.....(186*a*).

61. *Limits of C.* The solvability of the factorisants (in rational numbers) involves that

One or both of R, S must be +.....(187).

This involves the following limits of C in the four classes

C', C^{iv} may be any + numbers; or any − numbers $< \frac{1}{2}$, or > 1..(187*a*),

C''', C'''' may be any + numbers < 1; or any − numbers.....(187*b*).

62. *Possibility of factorisants.* This requires as a (preliminary) necessary condition—

Each of R, S must be a 2^{ic} Residue of the other.....(188).

Hence, (C^2-1) and therefore $(4C^2-4)$ should be a 2^{ic} Residue of $(2C+1)$.....(188*a*),

whence

3 should be a 2^{ic} Residue of every odd factor of $(2C+1)$.....(188*b*).

63. *Equivalence, Reciprocity.* Two characteristics (C_1, C_2) of the same or of different classes i., ii., iii., iv., and also the factorisants arising from them, viz.

$$R_1x^2+S_1y^2=z_1^2 \text{ (from } C_1); \quad R_2x^2+S_2y^2=z_2^2 \text{ (from } C_2),$$

will be said to be *equivalent* or *reciprocal* respectively, when they lead to the same Sextans (N), viz.

Equivalent with same x, y; *Reciprocal* with interchanged x, y.

The necessary and sufficient condition for this is

Equivalence, (same x, y); $R_1 : R_2 = S_1 : S_2 = z_1^2 : z_2^2$.....(189),

Reciprocity, (interchanged x, y); $R_1 : S_2 = S_1 : R_2 = z_1^2 : z_2^2$.....(190).

63*a*. *Equivalence.* In each of the four Classes the condition $R_1 : R_2 = S_1 : S_2$ gives

$$2C_1+1 : 2C_2+1 = C_1^2-1 : C_2^2-1.....(191),$$

* This is not absolutely necessary (though always convenient). If x, y be fractions, they can be made integers by multiplying by the L.C.M. of their denominators; (see Ex. 3 of Art. 80).

which—on reduction—leads to

$$(C_1 - C_2),\ \{C_1C_2 + \tfrac{1}{2}(C_1 + C_2) + 1\} = 0,$$

whence one solution is $C_1 = C_2 = 0$, (an identity).

The other solutions are given by

$$C_1C_2 + \tfrac{1}{2}(C_1 + C_2) + 1 = 0,$$

whence
$$C_2 = -\frac{C_1 + 2}{2C_1 + 1}, \quad C_1 = -\frac{C_2 + 2}{2C_2 + 1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (192),$$

and it is seen that to a given C_1 or C_2 there is always *one*, and *only one equivalent* characteristic C_2 or C_1; also that C_1, C_2 are *mutually equivalent*, and interchangeable. Also the only integer values are

$$(C_1,\ C_2) = (1,\ \bar{1}),\ (\bar{1},\ 1),\ (0,\ \bar{2}),\ (\bar{2},\ 0),$$

but there are an infinite number of fractional equivalent pairs.

63*b*. *Reciprocity.* In each of the four Classes the condition $R_1 : S_2 = S_1 : R_2$ gives

$$2C_1 + 1 : C_2^2 - 1 = C_1^2 - 1 : 2C_2 + 1,$$

or
$$(C_1^2 - 1)(C_2^2 - 1) = (2C_1 + 1)(2C_2 + 1).$$

Any given value of C_1 leads to a 2^{iv} equation in C_2, whence arise two values of C_2 which—as it turns out—are always rational, viz.

$$C_2 = \frac{C_1 + 2}{C_1 - 1},\ \text{or}\ \frac{-C_1}{C_1 + 1};\ \text{and}\ C_1 = \frac{-C_2}{C_2 + 1}\ \text{or}\ \frac{C_2 + 2}{C_2 - 1} \ldots\ldots (193),$$

Also these two values of C_2 will be found to satisfy the condition (192) of equivalence, and are therefore *mutually equivalent.*

63*c* *Characteristic Quartett.* Let C, C_σ, C_τ, C_ρ be four characteristics such that

C is equivalent to C_σ, and reciprocal to C_τ and C_ρ .....(194).

Then, by the condition (193) of reciprocity, it will be found that

Each of C, C_σ is reciprocal to each of C_τ, C_ρ, and *vice versâ*....(194*a*),

and, it has already been shewn (Art. 63*a*) that

C, C_σ are an Equivalent pair; and C_τ, C_ρ are an Equivalent pair....(194*b*).

Hence all possible characteristics may be arranged in sets of four related as above: such sets are styled *Characteristic*

Quartetts. Various other simple relations exist among the four members, *e.g.*

$$\frac{1}{C}+\frac{1}{C_\rho}=-1=\frac{1}{C_e}+\frac{1}{C_r} \quad \text{.....(194}c\text{)},$$

Excluding the case of two equal characteristics (which sometimes occur in same class, see Art. 63*d*), the conditions of equality and reciprocity (192), (193), appears to involve that the four members (C, C_e, C_r, C_ρ) of a Quartett should fall one in each of the four classes (i., ii., iii., iv. of Art. 58). Hence all Sextan factorisants may be grouped into sets of four, one in each of the four classes (i., ii., iii., iv.), with their characteristics forming a Quartett, and the four factorisants of such Quartett lead to the same Sextan Series.

As far as mere factorisation is concerned, any one characteristic of a Quartett suffices (as they all lead to the same Sextans); but, as the factorisants resulting are of different degrees of facility of solution, it is useful in practical factorising work to have the choice of any of the four.

63*d*. *Equal Characteristics in Quartett.* With the notation C, C_e, C_r, C_ρ, as in Art. 63*c*, it will be found that the only cases in which two of them can be equal—whether with same, or with opposite, sign—are

$$\left.\begin{matrix} C=C_\rho \\ C_e=C_r \end{matrix}\right\} \text{ give } \left\{\begin{matrix} C & C_e & C_r & C_\rho \\ \overline{2}, & 0, & 0, & \overline{2} \\ 0, & \overline{2}, & \overline{2}, & 0 \end{matrix}\right\}$$

$$\left.\begin{matrix} C=-C_e \text{ gives} \\ C_r=-C_\rho \text{ gives} \end{matrix}\right\} \left\{\begin{matrix} C & C_e & C_r & C_\rho \\ 1, & \overline{1}, & \infty, & -\frac{1}{2} \\ \overline{1}, & 1, & -\frac{1}{2}, & \infty \\ -\frac{1}{2}, & \infty, & \overline{1}, & 1 \\ \infty, & -\frac{1}{2}, & 1, & \overline{1} \end{matrix}\right\} \text{ Ineffective.}$$

63*e*. *Case of* $R+S=0$. This is one of the simplest cases. In each of the four classes (Art. 58) this leads to the same equation for C, viz.

$$(2C+1)+C^2-1=0 \quad \text{.....(195)},$$

which is satisfied by $C=0$ or -2. Now it will be found that

$$C'=0,\ C''=0,\ C'''=\overline{2},\ C^{iv}=\overline{2}, \text{ each give } x^2-y^2=z^2 \quad \text{.....(195}a\text{)},$$

which is the defining property of a *Dimorph Sextan* (Art. 41).

Also

$$C'=\overline{2},\ C''=\overline{2},\ C'''=0,\ C^{iv}=0, \text{ each lead to } x^2-y^2=\tfrac{1}{3}z^2 \text{ or } 3z^2 \quad \text{..(195}b\text{)},$$

which is the defining property of a *Trin-Aurifeuillian Sextan* (Art. 40).

Thus these two important kinds of factorisable Sextans are at once disclosed by the simplest application of the Diophantine Method.

64. *General Solutions.* The four factorisants may to some extent be treated of together under the single type

$$Rx^2 + Sy^2 = z^2 \; ; \; [R, S \text{ not both } -] \ldots\ldots (196).$$

Two simple Cases arise

$$\text{Case } 1^\circ. \quad R = \rho^2 ; \qquad \text{Case } 2^\circ. \quad S = \sigma^2 \ldots\ldots (197).$$

They admit of *general solution* in two ways in each case in terms of two arbitrary—but mutually prime—integers, (t, u).

Case 1°. $R = \rho^2$.

$$1^\circ a. \quad \rho x = t^2 - Su^2, \quad y = 2tu, \quad z = t^2 + Su^2 \ldots\ldots\ldots\ldots (198a),$$

$$1^\circ b. \quad \rho x = \tfrac{1}{2}(t^2 - Su^2), \; y = tu, \quad z = \tfrac{1}{2}(t^2 + Su^2) \ldots\ldots\ldots\ldots (198b).$$

Case 2°. $S = \sigma^2$.

$$2^\circ a. \quad x = 2tu, \quad \sigma y = t^2 - Ru^2, \qquad z = t^2 + Ru^2 \ldots\ldots\ldots\ldots (199a),$$

$$2^\circ b. \quad x = tu, \quad \sigma y = \tfrac{1}{2}(t^2 - Ru^2), \; z = \tfrac{1}{2}(t^2 + Ru^2) \ldots\ldots\ldots\ldots (199b).$$

65. *Serial Solutions.* When any one solution (x_r, y_r, z_r) of a factorisant is known, then writing the factorisant in the three forms

1. $z^2 - Sy^2 = Rx^2$; S being $+$, & $+\square$

2. $z^2 - Rx^2 = Sy^2$; R being $+$, & $+\square$

3. $Rx^2 + Sy^2 = z^2 \ldots\ldots (200)$, RS being $-$, & $+-\square \ldots (200a)$.

Two infinite series of solutions (x_r, y_r, z_r) may usually be obtained for the simple cases of

$$1^\circ. \; x = x_0 \text{ (const.)} ; \quad 2^\circ. \; y = y_0 \text{ (const.)} ; \quad z = z_0 \text{ (const.)} \ldots\ldots (200b),$$

respectively, when the above conditions as to R, S are fulfilled : (but these conditions can *only be satisfied in two* of the three Cases 1°, 2°, 3° at once). This is to be done by aid of the solutions (τ, υ)—here supposed known—of the associated Pellian equation, here styled the *Unit-form*.

$$1^\circ. \; \tau^2 - S\upsilon^2 = +1 ; \quad 2^\circ. \; \tau^2 - R\upsilon^2 = +1 ; \quad 3^\circ. \; \tau^2 - RS\upsilon^2 = +1 \ldots . (201).$$

The first application of the unit-form usually gives *two* distinct new solutions in each case of 1°, 2°, 3°, viz.

1°. $z_1 = \tau z_0 \mp S\upsilon y_0$, $y_1 = \upsilon z_0 \mp \tau y_0$, $x_1 = x_0$

2°. $z_1 = \tau z_0 \mp R\upsilon x_0$, $x_1 = \upsilon z_0 \mp \tau x_0$, $y_1 = y_0$

3°. $x_1 = \tau x_0 \pm S\upsilon y_0 \ldots (201a)$, $y_1 = R\upsilon x_0 \mp \tau y_0 \ldots (201b)$, $z_1 = z_0 \ldots\ldots\ldots (201c)$.

wherein the *same sign* (either both −, or both +) is to be used in any one pair of y_1, z_1 in Cases 1°, 2°, and opposite signs in Case 3°.

From *each* of these solutions (x_1, y_1, z_1) an infinite series of solutions (x_r, y_r, z_r) may now be obtained by the following "succession-formulæ,"

$$1^\circ.\ \begin{matrix} z_{r+1}=\tau z_r+Svy_r \\ y_{r+1}=\upsilon z_r+\tau y_r \\ x_{r+1}=x_r=x_0 \end{matrix} \quad \Big| \quad 2^\circ.\ \begin{matrix} z_{r+1}=\tau z_r+R\upsilon x_r \\ x_{r+1}=\upsilon z_r+\tau x_r \\ y_{r+1}=y_r=y_0 \end{matrix} \quad \Big| \quad 3^\circ.\ \begin{matrix} x_{r+1}=\tau x_r+Svy_r \ldots (202a), \\ y_{r+1}=R\upsilon x_r+\tau y_r \ldots (202b), \\ z_{r+1}=z_r=z_0 \ldots\ldots (202c). \end{matrix}$$

or, by the following, which give either series of x_r, y_r, z_r separately).

$$1^\circ.\ \begin{matrix} z_{r+1}=2\tau z_r-z_{r-1} \\ y_{r+1}=2\tau y_r-y_{r-1} \end{matrix} \quad \Big| \quad 2^\circ.\ \begin{matrix} z_{r+1}=2\tau z_r-z_{r-1} \\ x_{r+1}=2\tau x_r-x_{r-1} \end{matrix} \quad \Big| \quad 3^\circ.\ \begin{matrix} x_{r+1}=2\tau x_r-x_{r-1} \ldots (203a), \\ y_{r+1}=2\tau y_r-y_{r-1} \ldots (203b). \end{matrix}$$

Thus there are usually *two* infinite Series of solutions, arising from the double sign (±) in the formulæ (201*a*, *b*, *c*) for x_1, y_1, z_1 in each of the possible Cases of 1°, 2°, 3°. But, if the use of the − sign in these formulæ leads to

$$1^\circ.\ \begin{matrix} y_1=\upsilon z_0-\tau y_0 \\ =0, \text{ or } y_0 \end{matrix} \quad \Big| \quad 2^\circ.\ \begin{matrix} x_1=\upsilon z_0-\tau x_0 \\ =0, \text{ or } x_0 \end{matrix} \quad \Big| \quad 3^\circ.\ \begin{matrix} y_1=R\upsilon x_0-\tau y_0 \\ =0, \text{ or } y_0 \ldots (204), \end{matrix}$$

then the whole of the Series depending thereon disappears, leaving *only the one infinite Series* depending on the use of the + sign in the formulæ.

Solutions formed as above—*i.e.* with x, y, or z constant throughout the Series—will be styled *Serial solutions*, and the solution (x_0, y_0, z_0) from which they originate will be styled the *Base-Solution*.

65*a*. *Simple Serial Solutions.* The series obtained by the Rules of Art. 65 with

$$1^\circ.\ x_r=z_0=1\ ;\quad 2^\circ.\ y_r=y_0=1 \ldots\ldots\ldots\ldots (205),$$

giving *Simple* Sextans (Art. 1) are among the most interesting; they will be styled *Simple Serial Solutions*.

66. *Unit-forms.* The unit-forms (201), required to aid in forming the above serial solution, always admit of solution: but, in some cases the solution (τ, υ) is known algebraically: much work may often be saved by observing this.

Form 1° of (201). $\tau^2 - S\upsilon^2 = +1$; [with $S+$].

Class i. Here $S=C'^2-1$; then $\tau=C'$, $\upsilon=1$, $[C'>1] \ldots\ldots (206)$.

Class ii. Here $S=1-C''^2$; with $C''<1$.

In this case the factorisant, divided by C''^2 becomes

$$(z/C'')^2-(1/C''^2-1)y^2=-(2C''+1)(x/C'')^2 \quad \ldots\ldots\ldots(207),$$

whilst the unit-form becomes

$$\tau^2-(1/C''^2-1)v^2=+1\,; \quad \text{whence } \tau=1/C'',\ v=1 \quad \ldots\ldots(207a).$$

FORM 2° of (201). $\tau^2-Rv^2=+1$; [with $R+$].

CLASS i. Let $2C'=-\rho^2$, then $R=1-\rho^2$.

In this case the factorisant, divided by ρ^2, becomes

$$(z/\rho)^2-(1/\rho^2-1)x^2=S(y/\rho)^2 \quad \ldots\ldots\ldots\ldots\ldots(208),$$

whilst the unit-form becomes

$$\tau^2-(1/\rho^2-1)v^2=1\,; \quad \text{whence } \tau=1/\rho,\ v=1 \quad \ldots\ldots(208a).$$

CLASS ii. Let $2C''=-\rho^2$; then $R=\rho^2-1$, [$\rho>1$]

Here $$\tau=\rho,\ v=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(209).$$

67. *Chain-formation.* Serial Solutions of Class i. have the remarkable property of always yielding Sextan x-Chains (with x constant).

By (169, 185a), $$M_r=P_r+Q_r=(x_r^2+C'y_r^2)+y_rz_r, \ldots\ldots\ldots\ldots\ldots(210a),$$

$$L_{r+1}=P_{r+1}-Q_{r+1}=(x_{r+1}^2+C''y_{r+1}^2)-y_{r+1}z_{r-1} \ldots\ldots(210b).$$

And, by the Chain-property ($L_{r+1}=M_r$)

$$x_{r+1}^2+C''y_{r+1}^2-y_{r+1}z_{r+1}=x_r^2+C'y_r^2+y_rz_r \quad \ldots\ldots\ldots(210c).$$

67*a*. *x-Chains.* The condition for an x-solution ($x_r=x_0$) a constant) gives

$$y_{r+1}(C'y_{r+1}-z_{r+1})=y_r(C'y_r+z_r) \quad \ldots\ldots\ldots\ldots\ldots(211).$$

The factorisant i. (Art. 57) may be written

$$z_r^2-(C''^2-1)y_r^2=(2C''+1)x_0^2 \quad \ldots\ldots\ldots\ldots\ldots(212),$$

and its "unit-form" (Art. 66) has $\tau=C'$, $v=1$ .....(212a),

whence the serial "succession-formulæ," (Art. 65) give

$$y_{r+1}=z_r+C'y_r, \quad -y_r=z_{r+1}-C'y_{r+1} \quad \ldots\ldots\ldots\ldots\ldots(213),$$

and these values satisfy the above x-Chain condition identically. Hence any serial solution 1° of Art. 65 with $x_r=x_0$ (a constant) yields an x-Chain in Class i., the only condition being

$$C^2-1 \text{ should be } +, \text{ (or } C>1), \text{ and } +\square \quad \ldots\ldots\ldots(214).$$

The succession-formulæ are

Either $$z_{r+1}=C'z_r+(C'^2-1)y_r,\ y_{r+1}=z_r+C'y_r \quad \ldots\ldots\ldots\ldots(215a),$$

or $$z_{r+1}=2C'z_r-z_{r-1},\ y_{r+1}=2C'y_r-y_{r-1} \quad \ldots\ldots\ldots\ldots\ldots(215b),$$

and the factors (L_r, M_r) of N_r reduce to

$$L_r = \pm x_0^2 + y_{r-1} y_r, \quad M_r = \pm x_0^2 + y_r y_{r+1} \ldots\ldots\ldots\ldots (215c),$$

wherein x_0^2 has the same sign as C', (if y_{r-1}, y_r, y_{r+1} be taken +).

67*b*. *y-Chains.* The conditions for a y-serial solution, being $y = y_0$ (a constant), reduces the chain-condition (210*c*) to

$$x_{r+1}^2 - x_r^2 = y_0 \ (z_{r+1} + z_r).$$

But $$z_{r+1}^2 - z_r^2 = 2C' \ (x_{r+1}^2 - x_r^2).$$

Hence $$z_{r+1} - z_r = 2C\, y_0, \text{ (a const.)} \ldots\ldots\ldots (216),$$

so that the z_r are in arithmetical progression: but this condition seems difficult to realise.

[The y-Chains here discussed are of a different character to those formed in Art. 43*b*, 44*b*, 45*b*, although $y_r = y_0 =$ constant in all the Cases. In the present Case the y-symbol of N_r is that *to which the characteristic C is attached* in forming the factorisant].

68. *Suitable Characteristics.* To use this Diophantine process at all, the factorisants must be *solvables*. A method will now be developed of determining characteristics (C) which shall yield *solvable factorisants*. These will be styled *Suitable Characteristics*. It will be shown that every Sextan will serve to yield at least 12 suitable characteristics *for each way* in which it can be expressed in any of the 2^{ic} forms of type (166). The Sextans (N_0) used for this purpose will be styled the *Base-Sextans*.

Let $$N_0 = x_0^4 - x_0^2 y_0^2 + y_0^4 \ldots\ldots\ldots\ldots (217),$$

and let $$N_0 = P_0^2 - Q_0^2 = a_0^2 + b_0^2 = A_0^2 + 3B_0^2 = A_0'^2 + 3B_0'^2 \ldots\ldots (217a),$$

be any one set of the 2^{ic} forms of which it is capable.

Then, by (185*a*–*d*)

i. $\pm P_0 = x_0^2 + C' y_0^2, \quad Q_0 = y_0 z_0' \ldots\ldots\ldots\ldots (218),$
whence $C' = (\pm P_0 - x_0^2)/y_0^2 \ldots\ldots\ldots\ldots (218a).$

ii. $\pm a_0$ or $\pm b_0 = x_0^2 + C'' y_0^2, \quad b_0$ or $a_0 = y_0 z_0'' \ldots\ldots\ldots\ldots (219),$
whence $C'' = (\pm a_0 - x_0^2)/y_0^2$, or $= (\pm b_0 - x_0^2)/y_0^2 \ldots\ldots (219a).$

iii. $\pm A_0 = x_0^2 + C''' y_0^2, \quad B_0 = y_0 z_0''' \ldots\ldots\ldots\ldots (220),$
whence $C''' = (\pm A_0 - x_0^2)/y_0^2 \ldots\ldots\ldots\ldots (220a).$

iv. $\pm A_0' = x_0^2 + C^{iv} y_0^2, \quad B_0' = y_0 z_0^{iv} \ldots\ldots\ldots\ldots (221),$
whence $C^{iv} = (\pm A_0' - x_0^2)/y_0^2 \ldots\ldots\ldots\ldots (221a).$

Hereby the four 2^{ic} partitions (217*a*) used above have yielded two values of C', four of C'', two of C''', and two of C^{iv}. And as x_0, y_0 are interchangeable in the Base-Sextan form (217), and in all the Results following, a second set of

values of C', C'', C''', C^{iv}, may be evolved. Hence, in all, four values of C, eight of C', four of C'', and four of C''', have been found from the single set of four 2^{ic} forms used (217*a*).

The set of four 2^{ic} forms used above may be the algebraic forms (Art. 54), and in the case of a *prime* Base (N_0) they are the *only ones available*: but any set of the arithmetic forms similar to the above of which a composite Base-Sextan (N_0) is capable will suffice. And each different set of such forms will yield a different set of characteristics.

Also all the characteristics (C) found by the above process will be *suitable*, *i.e.* they will yield factorisants which are certainly solvable, because one solution (viz. the x_0, y_0, z_0 of the Base-Sextan) is known. The total number of factorisants obtainable from each set of four 2^{ic} forms of a given Base-Sextan (N_0) is thus seen to be 20.

69. *Associates, Primaries and Secondaries.* The set of characteristics, and the corresponding factorisants, arising from any one set of 2^{ic} forms of a Base-Sextan, will be styled *Associated Characteristics* and *Factorisants* respectively. Those arising from the algebraic sets of 2^{ic} forms will be styled *Primary*: those arising from any arithmetic set of 2^{ic} forms will be styled *Secondary*. Each set contains 20 Associates (Art. 68), but the number of Associates practically available for factorisation is usually less than 20 in each set as follows—

Some of the characteristics turn out *ineffective*.
Cases of equivalent pairs or of reciprocal pairs usually exist,

as will be shown below.

70. *Primary Characteristics.* It will now be shown that out of the set of 20 primary characteristics—

4 are always *ineffective*.....(222*a*),
1 is ineffective when $x_0 y_0 = \epsilon$ .....(222*b*),
2 reciprocal pairs exist when $x_0 y_0 = \epsilon$ .....(222*c*),
1 equivalent pair exists when $x_0 = 1$ and $y = \omega$....(222*d*).

The number of effective independent primary characteristics, and corresponding factorisants, is therefore reduced *in general* as follows—

13 when $x_0 y_0 = \epsilon$; 15 when $x_0 = 1$, and $y_0 = \omega$ . . . .(222*e*).

In special cases (*i.e.* for special values of x_0, y_0) these numbers may be further reduced; but no general Rules seem applicable.

70*a*. *Extra Primary Characteristics.* Each of the Automorphs of the (algebraic) partition $N_0 = A'^2_0 - 3B'^2_0$ (Art. 6) can be made to yield a set of four (primary) characteristics of Class iv. (*i.e.* C'^v). But these values of A'_0, B'_0 are usually so large as to be inconvenient for arithmetical work.

But, when the algebraic partition (A'_0, B'_0) is not the Base-form (Art. 6) *i.e.*

when $(x_0 - y_0)^2 < x_0 y_0$ or $x_0^2 + y_0^2 < 3x_0 y_0$ .....(223),

then the Base-form—(with *minimum* A'_0, B'_0)—is

$$N_0 = (2x_0^2 - 3x_0 y_0 + 2y_0^2)^2 - 3(x_0^2 - y_0^2)^2 = A_0'^2 - 3B_0'^2 \text{.....(224)}.$$

These (minimum) values of A'_0, B'_0 give an extra set of four primary characteristics, which are sometimes useful. They have not in general any simple relations to the 20 primary characteristics above: though in special cases one or more may be equivalent to a reciprocal to some of them.

70*b*. *Ineffective Primary Characteristics.* It becomes necessary here to introduce notation showing clearly the effect of the interchange of x_0, y_0, and also of a_0, b_0; thus,

Let $N_0 = x_0^4 - x_0^2 y_0^2 + y_0^4 = h^4 - h^2 k^2 + k^4 = k^4 - k^2 h^2 + h^4$ .....(225),

so that $x_0 = h$ or k, $y_0 = k$ or h,

whilst a_0, A_0, A_0', have the values

$a_0 = (h^2 - k^2)$ or $(k^2 - h^2)$, $A_0' = \pm(h^2 + k^2)$,

$A_0 = \pm(h^2 - \frac{1}{2}k^2)$ with k even, or $\pm(k^2 - \frac{1}{2}h^2)$ with h *even*,

or $= \pm\frac{1}{2}(h^2 + k^2)$ with $hk = \omega$,

whence, for the values of x_0, y_0, a_0, A_0, A'_0 in the scheme below the values of C'', C''', C'^v result, as shown

	x_0, y_0	a_0, A_0, A_0'	C	x_0, y_0	a_0, A_0, A_0'	C
ii	h, k	$a_0 = h^2 - k^2$	$C'' = -1$	k, h	$a_0 = k^2 - h^2$	$C'' = -1$..(226*a*)
iii	$h, k = \epsilon$	$A_0 = h^2 - \frac{1}{2}k^2$	$C''' = -\frac{1}{2}$	$k, h = \epsilon$	$A_0 = k^2 - \frac{1}{2}h^2$	$C''' = -\frac{1}{2}$..(226*b*)
iv	h, k	$A_0' = h^2 + k^2$	$C'^v = +1$	k, h	$A_0' = k^2 + h^2$	$C'^v = +1$.(226*c*).

and these characteristics lead to *degraded* factorisants as follows:

$C'' = 1$ gives $x^2 = z^2$; $C''' = -\frac{1}{2}$ gives $(\frac{1}{2}y)^2 = z^2$; $C'^v = +1$ gives $x^2 = z^2$,

and these factorisants are *ineffective* (for factorisation purposes) in that they merely reproduce the *algebraic* values a_0, A_0, A'_0 (instead of new arithmetic values). Hereby four out of the complete set of 20 primary characteristics, viz. two in each of the Classes ii., iv., are seen to be *ineffective*, and one more in Class iii. when $hk = \epsilon$.

70*c*. *Reciprocal and Equivalent Primary Characteristics.* Using the same notation (h, k) as in Art. (70*b*) the values of certain selected primary characteristics (C', C'', C''', C^{iv}) and of their reciprocals or equivalents are shown in the scheme below, with the data (P_0, a_0, A_0, A_0') which lead to them

	x_0, y_0	*Data.*	C	C_r, C_r, C_ρ	
ii	h, k	$a_0 = k^2 - h^2$	$C'' = (k^2 - 2h^2)/k^2$;	$C_r'' = (h^2 - \frac{3}{2}k^2)/h^2$	C'', C''' are
iii	$k = \epsilon$, h	$A_0 = h^2 - \frac{1}{2}k^2$	$C''' = (h^2 - \frac{3}{2}k^2)/h^2$;	$C_r''' = (k^2 - 2h^2)/k^2$	reciprl. (227*a*)
iii	$k = \epsilon$, h	$A_0 = \frac{1}{2}k^2 - h^2$	$C''' = -(h^2 + \frac{1}{2}k^2)/h^2$;	$C_r''' = -(2h^2 + k^2)/k^2$	C''', C^{iv} are
iv	h, k	$A_0' = -(h^2 + k^2)$	$C^{iv} = -(2h^2 + k^2)/k^2$;	$C_\rho^{iv} = -(h^2 + \frac{1}{2}k^2)/h^2$	reciprl. (227*b*)
i	$h = 1$, $k = \omega$	$P_0 = \frac{1}{2}(k^4 - k^2) + 1$	$C' = \frac{1}{2}(k^2 - 1)$;	$C_\epsilon' = -(k^2 + 3)/2k^2$	C', C'' are
iii	$h = 1$, $k = \omega$	$A_0 = \frac{1}{2}(1 + k^2)$	$C''' = -(k^2 + 3)/2k^2$;	$C_\epsilon''' = \frac{1}{2}(k^2 - 1)$	equivt. (227*c*)

The above shows that among the primary characteristics—

One C'' and one C''' are reciprocal when $k = \epsilon$ .....(228*a*),

One C''' and one C^{iv} are reciprocal when $k = \epsilon$ .....(228*b*),

One C' and one C''' are equivalent when $h = 1$, & $k = \omega$....(228*c*),

The above results are *general*. A complete examination of the whole set of 20 primary characteristics, and of their equivalents and reciprocals for the general case will be found* to show that there are *no more cases in general*: but for particular Base-Sextans (*i.e.* for particular values of x_0, y_0) there are often several such cases; but it seems impossible to reduce these to general Rules.

The number of effective independent primary characteristics is therefore to be reduced in general as follows :—

Reduced by two when $x_0 y_0 = \epsilon$; Reduced by one when $x_0 = 1$, & $y_0 = \omega$..(228*d*).

and sometimes by a greater number in special cases.

71. *Primary Characteristics of Simple Sextans.* Table IX. gives the complete set of 20 primary Characteristics arising from the Simple Base-Sextan

$$N_0 = \frac{x_0^6 + y_0^6}{x_0^4 + y_0^4} = \frac{1^6 + k^6}{1^2 + k^2} = 1 - k^2 + k^4.$$

together with the data

$x_0 = 1$ or k, $y_0 = k$ or 1; (P_0, Q_0), (a_0, b_0), (A_0, B_0), (A_0', B_0'),

from which they are derived by the rules of Art. 68.

It will be seen that there are

4 of C', 8 of C'', 4 of C''', 4 of C^{iv}.

Two sets of 4 of C''' are given; one set for $k = \epsilon$, one set for $k = \omega$. An "extra" set of 4 of C^{iv} (viz. Nos. 5, 6, 7, 8) is given for the Case described in Art. 70*a*, when the algebraic (A_0', B_0') is not of the Base-form.

* Cf. Tab. IX., Art. 71.

The Column headed $E\,R$ shows the ineffective cases and the equivalent and reciprocal primes as follow :—

I denotes that C_1'', C_4'', C_1''' (when $k=\varepsilon$), C_1^{iv} are *ineffective* (see Art. 70*b*).

E, *R*. Entries under these letters in any one line shew the particular C which is *equivalent* or *reciprocal* to the C of that line, (Art. 70*c*) ; thus

C_1', C_2''' ($k=\omega$) are *equivalent*.
C_2'', C_3''' ($k=\varepsilon$) are *reciprocal*.
C_4''' ($h=\varepsilon$), C_2^{iv} are *reciprocal*.

It will be seen that these C_2', C_3', C_4' cannot *in general* (since they involve k^4) be either equivalent or reciprocal to any other (primary) C; though they may be so for particular values of k.

TAB. IX.

Primary Characteristics (C) of Simple Sextans.

$N_0=(x_0{}^4-x_0{}^2y_0{}^2+y_0{}^4)=(1-k^2+k^4)$.

	x_0, y_0	P_0	Q_0	z_0	C'	E R
1	1, k	$\frac{1}{2}(k^4-k^2)+1$,	$\frac{1}{2}(k^4-k^2)$	$\frac{1}{2}(k^3-k)$,	$\frac{1}{2}(k^2-1)$	C_2'''
2	1, k	$-[\frac{1}{2}(k^4-k^2)+1]$,	$\frac{1}{2}(k^4-k^2)$	$\frac{1}{2}(k^3-k)$,	$-[\frac{1}{2}(k^4-k^2)+2]\ k^2$	
i. 3	k, 1	$\frac{1}{2}(k^4-k^2)+1)$,	$\frac{1}{2}(k^4-k^2)$	$\frac{1}{2}(k^4-k^2)$,	$\frac{1}{2}(k^4-3k^2)+1$	
4	k, 1	$-[\frac{1}{2}(k^4-k^2)+1]$,	$\frac{1}{2}(k^4-k^2)$	$\frac{1}{2}(k^4-k^2)$,	$-[\frac{1}{2}(k^4+k^2)+1]$	
	x_0, y_0	a_0	b_0	z_0	C''	
1	1, k	$1-k^2$,	k	1 ,	-1	I
2	1, k	k^2-1 ,	k	1 ,	$(k^2-2)\ k^2$	C_3'''
3	k, 1	$1-k^2$,	k	k ,	$1-2k^2$	
4	k, 1	k^2-1 ,	k	k ,	-1	I
ii.						
5	1, k	k ,	$1-k^2$	$(1-k^2)/k$,	$(k-1)/k^2$	
6	1, k	$-k$,	$1-k^2$	$(1-k^2)/k$,	$-(k+1)/k^2$	
7	k, 1	k ,	k^2-1	$(k^2-1)/k$,	$k-k^2$	
8	k, 1	$-k$,	k^2-1	$(k^2-1)/k$,	$-(k+k^2)$	
	x_0, y_0	A_0	B_0	z_0	C'''	
iii. 1	1, $k=\varepsilon$	$1-\frac{1}{2}k^2$,	$\frac{1}{2}k^2$	$\frac{1}{2}k$,	$-1,2$	I
2	1, $k=\varepsilon$	$\frac{1}{2}k^2-1$,	$\frac{1}{2}k^2$	$\frac{1}{2}k$,	$(\frac{1}{2}k^2-2)/k^2$	
($k=\varepsilon$) 3	$k=\varepsilon$, 1	$1-\frac{1}{2}k^2$,	$\frac{1}{2}k^2$	$\frac{1}{2}k^2$,	$1-\frac{3}{2}k^2$	C_2''
4	$k=\varepsilon$, 1	$\frac{1}{2}k^2-1$,	$\frac{1}{2}k^2$	$\frac{1}{2}k^2$,	$-(1+\frac{1}{2}k^2)$	C_2^{iv}
iii. 1	1, $k=\omega$	$\frac{1}{2}(1+k^2)$,	$\frac{1}{2}(1-k^2)$	$\frac{1}{2}(1-k^2)/k$,	$\frac{1}{2}(k^2-1)\ k^2$	
2	1, $k=\omega$	$-\frac{1}{2}(1+k^2)$,	$\frac{1}{2}(1-k^2)$	$\frac{1}{2}(1-k^2)/k$,	$-\frac{1}{2}(k^2+3)\ k^2$	C_1'
($h=\omega$) 3	$k=\omega$, 1	$\frac{1}{2}(k^2+1)$,	$\frac{1}{2}(k^2-1)$	$\frac{1}{2}(k^2-1)/k$,	$\frac{1}{2}(1-k^4)$	
4	$k=\omega$, 1	$-\frac{1}{2}(k^2+1)$,	$\frac{1}{2}(k^2-1)$	$\frac{1}{2}(k^2-1)/k$,	$-\frac{1}{2}(1+3k^2)$	
	x_0, y_0	A_0'	B_0'	z_0	C^{iv}	
1	1, k	$1+k^2$,	k	1 ,	1	I
2	1, k	$-(1+k^2)$,	k	1 ,	$-(k^2+2)/k^2$	C_4'''
iv. 3	k, 1	k^2+1 ,	k	k ,	1	I
4	k, 1	$-(k^2+1)$,	k	k ,	$-(2k^2+1)/k^2$	
iv. 5	1, k	$(2-3k+2k^2)$,	$1-2k+k^2$	$(1-2k+k^2)/k$,	$(1-3k+2k^2)\ k^2$	
Extra 6	1, k	$-(2-3k+2k^2)$,	$1-2k+k^2$	$(1-2k+k^2)/k$,	$-(3-3k+2k^2)\ k^2$	
7	k, 1	$(2-3k+2k^2)$,	$1-2k+k^2$	$(1-2k+k^2)$,	$(2-3k+k^4)$	
8	k, 1	$-(2-3k+2k^2)$,	$1-2k+k^2$	$(1-2k+k^2)$,	$-(2-3k+3k^2)$	

72. *Secondary Characteristics.* When N_0 is composite, then the set of four associate 2^{ic} forms

$$(P_0, Q_0),\ (a_0, b_0),\ (A_0, B_0)\ (A'_0, B'_0),$$

belonging to each pair of its cofactors ($N_0 = L_0 M_0$) gives rise to a set of 20 Secondary characteristics, and factorisants. The automorphs of the (A'_0, B'_0) form also each give rise to an extra set of four characteristics. These secondary characteristics are not usually all effective and independent; there are sometimes some ineffective and some equivalent or reciprocal pairs in each set; but no general Rules seem applicable. Also they do not seem to be related in any simple way to the primary characteristics.

[See Art. 73]. TAB X.

	$N_0 = 13 = 1^4 - 1^2.2^3 + 2^4$							
	$x_0\ y_0$	$P_0\ Q_0$	z_0	C'	E	R	R	I
1	1, 2	7, 6	3	3/2,				
2	1, 2	−7, 6	3	−2 , 0 , 0 , −2				T
i. 3	2, 1	7, 6	6	3 ,			−3/4	
4	2, 1	−7, 6	6	−11 ,		3/4 ,		
	$x_0\ y_0$	$a_0\ b_0$	z_0	C''	E	R	R	
1	1, 2	−3, 2	1	−1 , 1 , −1/2,				I
2	1, 2	3, 2	1	1/2, −5/4, −5 ,				
3	2, 1	−3, 2	2	−7 ,				
4	2, 1	3, 2	2	−1 , 1 , −1/2,				I
ii.								
5	1, 2	2, 3	3/2	1/4, −3/2, −3 ,				
6	1, 2	−2, 3	3/2	−3/4,			3	
7	2, 1	2, 3	3	−2 , 0 , 0 , −2				T
8	2, 1	−2, 3	3	−6 ,				
	$x_0\ y_0$	$A_0\ B_0$	z_0	C'''	E	R	R	
1	1, 2	−1, 2	1	−1/2,	−1 ,	1		I
2	1, 2	1, 2	1	0 , −2 , −2 ,		0		T
iii. 3	2, 1	−1, 2	2	−5 ,	1/2 ,	−5/4		
4	2, 1	1, 2	2	−3 ,	1/4 ,	−3/2		
	$x_0\ y_0$	$A_0'\ B_0'$	z_0	C'^v	E	R	R	
1	1, 2	5, 2	1	1 , −1 , −1/2,				I
2	1, 2	−5, 2	1	−3/2, 1/4, −3,				
3	2, 1	5, 2	2	1 , −1 , −1/2,				I
4	2, 1	−5, 2	2	−9 ,				
iv.								
5	1, 2	4, 1	1/2	3/4,	−11 ,			
6	1, 2	−4, 1	1/2	−5/4, 1/2,			−5	
7	2, 1	4, 1	1	0 , −2 , −2 ,		0		T
8	2, 1	−4, 1	1	−8 ,				

TAB. X. (*continued.*)

[See Art. 73].

	$N_0 = 61 = 3^4 - 3^2.2^2 + 2^4$							
	$x_0 y_0$	P_0 Q_0	z_0	C'	E	R	R	I
1	3, 2	31, 30	15	11/2,				
2	3, 2	−31, 30	15	−10 ,			−10/9	
i. 3	2, 3	31, 30	10	3 ,			−3/4	
4	2, 3	−31, 30	10	−35/9,				
	$x_0 y_0$	a_0 b_0	z_0	C''	E	R	R	
1	3, 2	5, 6	3	−1 , 1 .	−1/2			I
2	3, 2	−5, 6	3	−7/2, −1/4,	1/3			
3	2, 3	5, 6	2	1/9,				
4	2, 3	−5, 6	2	−1 , 1 ,	−1/2			I
ii.								
5	3, 2	6, 5	5/2	−3/4 ,			3	
6	3, 2	−6, 5	5/2	−15/4,				
7	2, 3	6, 5	5/3	2/9 ,				
8	2, 3	−6, 5	5/3	−10/9,			−10	
	$x_0 y_0$	A_0 B_0	z_0	C'''	E	R	R	
1	3, 2	7, 2	1	−1/2 ,	−1 ,	1		I
2	3, 2	−7, 2	1	−4 ,			−4/3	
iii. 3	2, 3	7, 2	2/3	1/3 ,	−7/2,	−1/4		
4	2, 3	−7, 2	2/3	−11/9,			−11/2	
	$x_0 y_0$	A_0' B_0'	z_0	C'^{v}	E	R	R	
1	3, 2	13, 6	3	1 , −1,	−1/2,			I
2	3, 2	−13, 6	3	−11/2,			−11/9	
3	2, 3	13, 6	2	1 , −1 ,	−1/2,			I
4	2, 3	−13, 6	2	−17/9,				
iv.								
5	3, 2	8, 1	1/2	−1/4 ,	−7/2,		1/3	
6	3, 2	−8, 1	1/2	−17/4,				
7	2, 3	8, 1	1/3	4/9 ,				
8	2, 3	−8, 1	1/3	−4/3 ,			−4	

73. *Characteristics, Examples.* The Tables—(No. X. above and XI. below)—give the complete set of 20 characteristics, viz.

4 of C', 8 of C'', 4 of C''', 4 of C'^{v}; and 4 extra of C'^{v} (when possible),

derived from the three small Base-Sextans

$$N_0 = 13\,; \quad N_0 = 61\,; \quad N_0 = 1261 = 13.97\,;$$

together with the complete set of data and results, as in Art. 71.

The columns of E, R show the numerical values of the characteristics *equivalent* (E) and *reciprocal* (R) to that in same line of the C column; (but these are given only* when found elsewhere also in the C column).

These Tables illustrate nearly all the properties of associate sets of characteristics developed in Art. 69—72.

$N_0 = 13$ & 61—being prime—have *only the primary* system (Art. 68).

$N_0 = 1261 = 13.97$—being composite—has *one primary* and *one secondary* system (Art. 69).

The primary systems have 5 *ineffectives*, marked I, (Art. 70*b*).

The secondary system has *no ineffectives* (Art. 72).

All the systems (k being *even*) have at least 2 reciprocal pairs (besides those due to "ineffectives"): any excess over this number is not a general property, but a specialty of the bases $N_0 = 13$, 61, &c. (Art. 70*c*).

The equivalent pairs (other than those due to "ineffectives") are not due to a general property (k being *even*), but are a specialty of the bases $N_0 = 13$, 61, &c.

It will be seen that the number of *effective* and *independent* characteristics is usually a good deal under 20.

TAB. XI.

	$N_0 = 1261$ (*Primary*)						$N_0 = 13.97$ (*Secondary*)		
	$x_0\ y_0$	$P_0\ Q_0$	C'	E	R	I	$P_0\ Q_0$	C'	R
1	1, 6	631, 630	35/2,				55, 42	3/2,	7
2	1, 6	−631, 630	−158/9,				−55, 42	−14/9,	
i. 3	6, 1	631, 630	595,				55, 42	19,	7/6
4	6, 1	−631, 630	−667,				−55, 42	−91,	
	$x_0\ y_0$	$a_0\ b_0$	C''	E	R		$a_0\ b_0$	C''	R
1	1, 6	35, 6	17/18,		−53		19, 30	1/2,	−5
2	1, 6	−35, 6	−1,	1,	−1/2	I	−19, 30	−5/9,	
3	6, 1	35, 6	−1,	1,	−1/2	I	19, 30	−17,	5/6
4	6, 1	−35, 6	−71				−19, 30	−55,	
ii. 5	1, 6	6, 35	5/36,				30, 19	29/36,	
6	1, 6	−6, 35	−7/36,				−30, 19	−31/36,	
7	6, 1	6, 35	−30,				30, 19	−6,	
8	6, 1	−6, 35	−42,				−30, 19	−66,	
	$x_0\ y_0$	$A_0\ B_0$	C'''	E	R		$A_0\ B_0$	C'''	R
1	1, 6	17, 18	4/9,				31, 10	5/6,	−17
2	1, 6	−17, 18	−1/2,		−1,1	I	−31, 10	−8/9,	
iii. 3	6, 1	17, 18	−19,		−19/18		31, 10	−5,	1/2
4	6, 1	−17, 18	−53,		17/18		−31, 10	−67,	
	$x_0\ y_0$	$A'_0\ B'_0$	C^{iv}	E	R		$A_0'\ B'_0$	C^{iv}	R
1	1, 6	37, 6	1,	−1,	−1/2	I	43, 14	7/6,	19
2	1, 6	−37, 6	−19/18,		−19		−43, 14	−11/9,	
3	6, 1	37, 6	1,	−1,	−1/2	I	43, 14	7,	3/2
4	6, 1	−37, 6	−73,				−43, 14	−79,	

* Equivalent and Reciprocal Characteristics exist of course *for every entry* in the C column, but are not entered here unless found also elsewhere in the C column; (thus shewing these values of C to be *not independent*.)

74. *Factorisants, Examples.* It will be well now to examine one complete set of Factorisants, say those arising from the smallest Sextan $N_0 = 13$ as Base. The full number of characteristics tabulated for $N_0 = 13$ in Art. 73 (Tab X.) is 24. Out of these, however, 15 may be rejected as follows :—

5 (marked *I*) are *ineffective*.

4 (marked *T*) lead to *Trin-Aurifexillians*, a case fully developed in Art. 40—40*b*.

6 (detailed below) are either *equivalent* or *reciprocal* to one tabulated as stated below :—

	Equivalent.	*Reciprocal.*
Rejected	C_2^{iv}, C_6^{iv}	C_6'', C_3''', C_4''', C_5^{iv}.
Tabulated	C_5', C_2''	C_3', C_2'', C_5'', C_4'.

This leaves only 9 *effective* and *independent* characteristics worth further examination, viz.

3 of C', 4 of C'', 2 of C^{iv} :

The Scheme below shows the 9 factorisants arising from the above 9 characteristics, together with the data from which they arise brought forward from Tab. X. of Art. 73.

The column headed *Serial* shows the sort of *Serial solution* possible, *i.e* (whether x, y, or z constant), thus :—

A single x, y, z in this column shows that only a single x-, y-, or z-solution exists; the entry x, x; y, y; z, z; shows that a double x-, y-, or z- solution exists.

Ref. No.	x_0 y_0	P_0 Q_0	z_0	C'	*Factorisant*	*Serial*
1	1, 2	7, 6	3	3/2	$4x^2 + \frac{5}{4}y^2 = z^2$	x,
i. 3	2, 1	7, 6	6	3	$7x^2 + 8y^2 = z^2$	x, x; y, y
4	2, 1	−7, 6	6	−11	$-21x^2 + 120y^2 = z^2$	x, x; z, z
	x_0 y_0	a_0 b_0	z_0	C''		
2	1, 2	3, 2	1	1/2	$-2x^2 + \frac{5}{2}y^2 = z^2$	x ; z
ii. 3	2, 1	−3, 2	2	−7	$13x^2 - 45y^2 = z^2$	y, y ; z
5	1, 2	2, 3	3/2	1/4	$-\frac{3}{2}x^2 + \frac{15}{16}y^2 = z^2$	x ; z
8	2, 1	−2, 3	3	−6	$11x^2 - 35y^2 = z^2$	y, y ; z
	x_0 y_0	A_0' B_0'	z_0	C^{iv}		
iv 4	2, 1	−5, 2	2	−9	$-\frac{17}{5}x^2 + \frac{9}{5}y^2 = z^2$	x, x; z
8	2, 1	−4, 1	1	−8	$-5x^2 + 21y^2 = z^2$	x, x; z

75. *Simple general Case.* One of the simplest cases algebraically factorisable arises from the Simple Base-Sextan

$$N_0 = \frac{1^6 + y_0^6}{1^2 + y_0^2} = \left(\frac{y_0^4 - y_0^2}{2} + 1\right)^2 - \left(\frac{y_0^4 - y_0^2}{2}\right)^2 = P_0^2 - Q_0^2 \ldots\ldots\ldots (229).$$

Here one characteristic (C'), and its factorisant are given at once by

$$C'=\tfrac{1}{2}(y_0^2-1),\quad z^2-(C'^2-1)y^2=y_0^2.1^2 \ldots\ldots\ldots\ldots(230),$$

which falls under Form 1° of Art. 65 (S being always +, and $x=1$ a constant). Hence, with the unit-form

$$\tau^2-S.v^2=+1,\ \text{where}\ \tau=C',\ S=C'^2-1,\ v=1 \ldots\ldots\ldots\ldots(231),$$

this always gives a *single* x-Series, which is in fact (Art. 67*a*) a *single* x-chain. Hence the succession-formulæ are (Art. 67*a*) either set of

$$z_{r+1}=C'.z_r+(C'^2-1)y_r,\ y_{r+1}=z_r+C'y_r,\qquad x_{r+1}=x_0=1 \ldots\ldots(232a),$$

$$z_{r+1}=2C'.z_r-z_{r-1},\qquad y_{r+1}=2C'y_r-y_{r-1},\ x_{r+1}=x_0=1 \ldots\ldots(232b),$$

and the Diophantine Factors (L_r, M_r) of N_r can be found at once from the y_r-series above by the formulæ

$$M_{r-1}=L_r=1+y_{r-1}y_r,\quad M_r=1+y_ry_{r+1}=L_{r+1} \ldots\ldots\ldots\ldots(232c).$$

It is now easy to write down a few of the values of y_r, z_r explicitly in terms of y_0. *E.g.*, those of y_r (as far as y_4), are

$r=0$, 1, 2, 3, 4, 5

$y_r=y_0$, $(y_0^3-y_0)$, $(y_0^5-2y_0^3)$, $(y_0^7-3y_0^5+y_0^3+y_0)$, $(y_0^9-4y_0^7+3y_0^5+2y_0^3-y_0)$, &c...(233),

but, in numerical work, it is generally more convenient to compute the y_r numerically from the succession-formulæ (232*a*, *b*), and to compute the successive L_r, M_r from (232*c*).

Ex. The Table XII. shows a few of the results of the above formulæ obtained from each of the Simple Base-Sextans defined by $y_0=2, 3, 4, \ldots 10$.

The line of x, y shows the Sextan elements (x_r, y_r) of the successive Sextans $N_r=(1+y_r^6)/(1+y_r^3)$, $[x_r=x_0=1$ throughout$]$.

The line of L, M shows the Diophantine Factors (L_r, M_r) of each N_r, arranged in such a way as to exhibit clearly the x-Chain property ($M_r=L_{r+1}$).

75*a*. *Generalized Formulæ* (of *Art.* 75). Changing y_r into y_r/x_r, and multiplying by such powers of x_r as just suffice to remove fractions, and render each result homogeneous, the Sextan takes the general form

$$N_r=(x_r^6+y_r^6)\div(x_r^2+y_r^2),\ \text{wherein}\ x_r=x_0=\textit{constant},$$

and y_r is given by

$r=0$, 1, 2, 3, 4

$y_r=y_0$, $(y_0^3-x_0^2y_0)$, $(y_0^5-2x_0^2y_0^3)$, $(x_0^7-3x_0^2y_0^5+x_0^4y_0^3+x_0^6y_0)$, &c.....(234).

[see Art. 75]. TAB. XII.

	$y_0=2,\ C'=3/2$						
$x,\ y$	1,2	1,6	1,16	1,42	1,110	1,288	1,754	1,1974	1,5168
L	1;	13;	97;	673;	4621;	13.2437;	37.5869;	73.20389;	13.277.2833;
M	13;	97;	673;	4621;	13.2437;	37.5869;	73.20389;	13.277.2833;	69923041;
	$y_0=3,\ C'=4$						
$x,\ y$	1,3		1,24		1,189	1,1488	1,11715	1,92232	
L	1;		73;		13.349;	281233;	13.37.36241;	5281.204601;	
M	73;		13.349;		281233;	13.37.36241;	5281.204601;		
	$y_0=4,\ C'=15/2$						
$x,\ y$	1,4		1,60		1,896	1,13380	1,199804		
L	1;		241;		37.1453;	37.457.709;	61.97.193.2341;		
M	241;		37.1453;		37.457.709;	61.97.193.2341;			
	$y_0=5,\ C'=12$						
$x,\ y$	1,5		1,120		1,2875	1,68880	1,1697245		
L	1;		601;		345001;	13.109.139753;	37.457.1993.3373;		
M	601;		345001;		13.109.139753;	37.457.1993.3373;			
	$y_0=6,\ C'=35/2$						
$x,\ y$	1,6		1,210		1,7344	1,256830			
L	1;		13.97;		109.14149;	6961.270961;			
M	13.97;		109.14149;		6961.270961;	13.157.			
	$y_0=7,\ C'=24$						
$x,\ y$	1,7		1,336		1,16121	1,773472			
L	1;		13.181;		157.34501;	1621.7692253;			
M	13.181;		157.34501;		1621.7692253;				
	$y_0=8,\ C'=63/2$						
$x,\ y$	1,8		1,504		1,31744	1,1999368			
L	1;		37.109;		15998077;				
M	37.109;		15998077;		13.488214.9061*				
	$y_0=9,\ C'=40$						
$x,\ y$	1,9		1,720		1,57591	1,4666560			
L	1;		6481;		41465521;				
M	6481;		41465521;		2652963966961*				
	$y_0=10,\ C'=99/2$						
$x,\ y$	1,10		1,990		1,98010	1,9701010			
L	1;		9901;		13.7463077;	5521.172196881;			
M	9901;		13.7463077;		5521.172196881;				

76. *Examples, Serial, Tab. XIII.* Here follow Serial Solutions of 8 of the 9 Factorisants of $N_0=13$, from the Table of Art. 74, viz. those arising from the 8 characteristics

$$C'=3,\ \overline{11};\quad C''=\tfrac{1}{2},\ \overline{7},\ \tfrac{1}{4},\ \overline{6};\quad C'''=\overline{9},\ \overline{8}.$$

[The first factorisant (with $C'=\frac{3}{2}$) of that Table is omitted here, examples thereof being given in Art. 75*a*].

* These large numbers have no factors <33000.

The *head-line* of each Series gives:

(1) The *reference-number*. (2) The *characteristic* (C). (3) The *factorisant*. (4) The *unit-form*. (5) The *value of the constant* x or y.

The factorisant is exhibited in the form adopted to an x- or y- solution (Art. 65), and the x- or y- Series alone are worked out: those with z constant (being of less interest), are omitted.

The co-factors L_r, M_r are ranged in separate lines so as to exhibit distinctly the chain-property (which occurs only in the x-Series of Class i.)

In Class i. ($C=3$, $\overline{11}$) the values of P, Q, from which x, y may be found are omitted, as not being of much interest. In the x-chains, in fact, z, P, Q are not required for mere factorisation; as, once the chain is started, the values of y_r, L_r, M_r are more easily found from the succession-formulæ.

In Classes ii. and iv. the 2^{ic} partitions (a, b), (a, b), (a, β), (a', β'), or (A', B'), (A', B'), (γ, δ), (γ', δ') of N_r L_r, M_r are given, being necessary for working out the L_r, M_r; and to help in tracing out the relations of the 2^{ic} parts of successive L_r, M_r which are often interesting.

In Ser. ii.—(2). $\alpha_r' = 3\alpha_r$, $\beta_r' = 3\beta_r - 2$ (if $r = \epsilon$), $\beta_r' = 3\beta_r + 2$ (if $r = \omega$).

In Ser. ii.—(5). $\alpha_{r+1} = \beta_r$, $\alpha'_{r+1} = \beta_r'$.

In Ser. ii.—(8). $\alpha_r \sim \beta_r = 1$, $\alpha_r' \sim \beta_r' = 5$.

In Ser. iv.—(4). $\gamma_r' = 5\gamma_r$, $\delta_r' = 5\delta_r \mp z$.

TAB. XIII.

i.—(3); $C=3$; $z^2 - 2(2y)^2 = 7x^2 = 7.2^2$; $3^2 - 2.2^2 = +1$; $x = x_0 = 2$.

x-Chn. 1°	x, y	2,1	2,9	2,53	2,369	2,1801
	L	1;	13;	13.37;	16381;	556513;
	M	13;	13.37;	16381;	556513;	18905101;
x-Chn. 2°	x, y	2,3	2,19	2,111	2,647	2,3771
	L	1;	61;	2113;	71821;	97.25153;
	M	61;	2113;	71821;	97.25153;	13.73.87337;

i.—(3); $C=3$; $z^2 - 7x^2 = 8y^2 = 8$; $8^2 - 7.3^2 = +1$; $y = y_0 = 1$.

y-Ser.	x, y	2,1	34,1	512,1	8638,1
	z	6	90	1434	22854
	L	1;	1069;	241.1213;	13.409.14029;
	M	13;	1249;	295201;	13.73.78649;

i.—(4); $C=-11$; $z^2 - 30(2y)^2 = -21x^2 = -21.2^2$; $11^2 - 30.2^2 = +1$; $x = x_0 = 2$.

x-Chn. 1°	x, y	2,1	2,17	2,373	*x-Chn.* 2°	2,5	2,109	2,2393
	z	6	186	4086		54	1194	26214
	L	1;	13;	6337;		1;	541;	97.2689;
	M	13;	6337;	13.234961;		541;	97.2689;	13.

ii.—(2); $C' = \frac{1}{2}$; $z^2 - 3(\frac{1}{2}y)^2 = -2x^2 = -2.1^2$; $2^2 - 3.1^2 = +1$; $x = x_0 = 1$.

x-Series	x, y	1,2	1,6	1,22	1,82	1,306	1,1142
	z	1	5	19	71	265	989
	a, b	3,2	35,6	483,22	6723,82	93635,306	1304163,1142
	a, b	3,2	19,30	213,418	3363,5822	46819,81090	652083,1129438
	a, β	1,0	3,2	11,6	41,24	153,88	571,320
	a', β'	3,2	9,4	33,20	123,70	459,266	1713,988
	L	1;	13;	157;	37.61;	3113;	13.33457;
	M	13;	97;	1489;	20029;	13.21649;	3910513;

TAB. XIII. (*continued*).

ii.—(3); $C''=-7$; $z^2-13x^2=-3.(4y)^2=-3.(4.1)^2$; $649^2-13.180^2=+1$; $y=y_0=1$.

y-Series 1°	$x,\ y$	2,1	938,1	*y-Series* 2°		1658,1
	z	2	3382			5978
	a, b	3,2	879843,938			2748963,1658
	$a,\ b$	3,2	879837,3382			2748957,5978
	$\alpha,\ \beta$	1,0	1,720			1,720
	α',β'	3,2	3,1222			3,3818
	L	1;	13.39877;			13.39877;
	M	13;	1493293;			14577133;

TAB. XIII*a*.

ii.—(5); $C''=1/4$; $z^2-15(\frac{1}{4}y^2=-\frac{3}{4}x^2=-\frac{3}{4}.1^2$; $4^2-15.1^2=+1$; $x=x_0=1$.

x-Series	$x,\ y$	1,2	1,14	1,110	1,866	1,6818
	$2z$	3	27	213	1677	13203
	a, b	2,3	14,195	110,12099	866,749955	6818,46485123
	$a,\ b$	2,3	50,189	3026,11715	187490,726141	11621282,45009027
	$\alpha,\ \beta$	0,1	1,6	6,49	49,384	384,3025
	α',β'	2,3	3,32	32,243	243,1922	1922,15123
	L	1;	37;	2437;	277.541;	13.715237;
	M	13;	1033;	13.4621;	3753133;	193.1204141;

ii.—(8). $C''=-6$; $z^2-11x^2=-35y^2=-35.1^2$; $16^2-11.5^2=+1$; $y=y_0=1$.

y-Series 1°	$x,\ y$	2,1	29,1	578,1	*y-Series* 2°	11,1	218,1	4349,1
	z	3	96	1917		36	723	14421
	a, b	2,3	840,29	334083,578		120,11	47523,218	18913800,4349
	$a,\ b$	2,3	96,835	1917,334078		36,115	723,47518	14424,18913795
	$\alpha,\ \beta$	1,0	13,12	249,250		3,2	51,50	1007,1008
	α',β'	3,2	31,36	667,672		21,26	473,468	9389,9384
	L	1;	313;	13.61.157;		13;	5101;	97.20929;
	M	13;	37.61;	37.24229;		1117;	442753;	13.13554829;

iv.—(4); $C''=-9$; $z^2-\frac{1}{3}(4y)^2=-\frac{1}{3}x^2=-\frac{1}{3}.2^2$; $3^2-\frac{8}{3}.0^2=+1$; $x=z_0=2$.

x-Series 1°	$x,\ y$	2,1	2,43	*x-Series* 2°		2,19
	z	2	222			98
	A', B'	5,2	1853,86			365,38
	$A',\ B'$	5,2	16637,9546			3245,1862
	$\gamma,\ \delta$	1,0	43,22			19,10
	$\gamma',\ \delta'$	5,2	215,112			95,48
	L	1;	397;			61;
	M	13;	13.661;			2113;

iv.—(8); $C''=-8$; $z^2-21y^2=-5x^2=-5.2^2$; $55^2-21.12^2=+1$; $x=x_0=2$.

x-Series 1°	$x,\ y$	2,1	2,67	*x-Series* 2°		2,43
	z	1	307			197
	A', B'	4,1	4493,134			1853,86
	$A',\ B'$	4,1	10109*,5230			4163*,2154
	$\gamma,\ \delta$	1,0	49,18			33,11
	$\gamma',\ \delta'$	4,1	149,52			94,35
	L	1;	1429;			661;
	M	13;	73.493;			13.397;

* see Art. 566 (page 3).

SEXTANS ; [2^{ic} *Partitions of L, M*].

[see Art. 77 following]. TAB. XIV.

i.—(1)	L	1 ;	13 ;	97 ;	673 ;	4621 ,	13.2437 ;	37.5869 ;
	a, b	1,0	3,2	9,4	23,12	61,30	159,80	417,208
	A, B	1,0	1,2	7,4	25,4	17,38	127,72	249,72
	A'B'	2,1	4,1	10,1	26,1	68,1	178,1	466,1

i.—(3) *x-Chain 1°*	L	1;	13;	13,37;	16381;	556513;	18905101;	*x-Chain 2°*	1 ; 61; 2113; 71821; 97.25153;
	a, b	1,0	3,2	15,16	91,90	527,528	3075,3074		1,0 5,6 33,32 189,190 1105,1104
	A, B	1,0	1,2	7,12	41,70	239,408	1393,2378		1,0 7,2 41,12 239,70 1393,408
	A'B'	2,1	4,1	22,1	128,1	746,1	4348,1		2,1 8,1 46,1 268,1 1562,1

i.—(3)	L	1 ; 1069;	241.1213;	13.409.14029;	M	13; 1249;	295201;	13.73.78649;
	a, b	1,0 13,30	222,493	7872,3553	a, b	3,2 15,32	224,495	7874,3555
	A, B	1,0 31,6	511,102	8161,1632	A, B	1,2 7,20	103,308	1633,4898
	A'B'	2,1 37,10	571,106	9025,1512	A'B'	4,1 41,12	623,176	9232,1879

i.—(4) *x-Chain 1°*	L	1;	13;	6337;	13.234961 ;	*x-Chain 2°*	1 ;	541 ; 97.2689 ;
	a, b	1,0	3,2	71,36	1563,782		1,0	21,10 457,228
	A, B	1,0	1,2	23,44	505,966		1,0	23,2 505,44
	A'B'	2,1	4,1	97,32	2140,713		2,1	28,9 625,208

ii.—(2)	L	1; 13;	157;	37.61;	3113; 13.33457;	M	13; 97;	1489; 20029; 12.21649;
	a, b	1,0 3,2	11,6	41,24	153,88 571,330	a, b	3,2 9,4	33,20 123,70 459,266
	A, B	1,0 1,2	7,6	23,24	89,88 329,330	A, B	1,2 7,4	17,20 73,70 263,266
	A'B'	2,1 4,1	13,2	55,16	185,32 692,121	A'B'	4,1 10,1	41,8 221,98 557,98

ii.—(5)	L	1; 37;	2437;	277.541;	13.715237;	M	13; 1033;	13.4621; 3753133;
	a, b	0,1 1,6	6,49	49,384	384,3025	a, b	2,3 3,32	32,243 243,1922
	A, B	1,0 5,2	43,14	335,112	2641,880	A, B	1,2 29,8	211,72 1679,558
	A'B'	2,1 7,2	55,14	433,112	3409,880	A'B'	4,1 35,8	275,72 2165,558

ii.—(8) *y-Series 1°*	L	1 ;	313 ;	13.61.157 ;		M	13; 37.61;	37.21229 ;
	a, b	1,0	13,12	249,250		a, b	3,2 31,36	667,672
	A, B	1,0	11,8	349,30		A, B	1,2 47,4	901,168
	A'B'	2,1	19,4	283,31		A'B'	4,1 55,16	1010,203

y-Series 2°	L	13 ;	5101 ;	97.20929 ;		M	13.1117 ; 442753 ;	13.135554829 ;
	a, b	3,2	51,50	1007,1008		a, b	21,26 473,468	9389,9384
	A, B	1,2	49,30	1369,228		A, B	61,60 191,368	
	A'B'	2,1	76,15	1519,304		A'B'	122,11 710,143	

77. 2^{ic} *Partitions of Co-factors L, M.* It might be expected that the 2^{ic} partitions of the co-factors L, M of the successive Sextans of a Serial Solution would be closely connected. In order to exhibit this clearly the 2^{ic} partitions of the L, M of a selected number of the Serial Solutions of Tables XII., XIII. are given in the Table (XIV.) above.

The reference-numbers (i.—1, i.—3, &c.) on the left are those of the Series in Tab. XII., XIII. here brought forward. The arrangement is similar to that of Tab. XIII. Each Series has a band to itself. The 2nd,

3rd, and 4th lines give the 2^{ic} partitions of the L-, M-factors shewn in the top line.

In Chain-Series (i. −1, i. −3, i. −4) the $M_r = L_{r+1}$ throughout, and the L-factors only are shown. In non-Chain Series the L-factors are shown in the left half, and M-factors in the right half of the band.

From these data it is possible (and in some cases* easy) to form formulæ for the relations between the "2^{ic} parts" of the successive L_r or M_r: but those formulæ would be *special to each Series*, and not applicable generally, so it is not thought worth while tracing them further.

78. *Allied* 6-*tan form, Diophantines.* The Diophantine Method of this Chapter may be also readily applied to Sextans, written in the *Allied Form* of Art. 25,

$$N = x^4 + 14x^2y^2 + y^4 \ldots\ldots\ldots\ldots\ldots\ldots (235),$$

the (algebraic) 2^{ic} partitions of which are given in (52*a*, *b*).

But, to gain the full advantage of this Method, the "working conditions" (3*b*), (51*b*) of N always odd must be relaxed. Two alternative transformations are required, thus

i. $x' = y - x$, $y' = y + x$; $x = \frac{1}{2}(y' - x')$, $y = \frac{1}{2}(y' + x')$
require $x'y' = \omega$, $xy = \varepsilon$, and give $N = \Omega = N_{vi} \ldots\ldots\ldots\ldots (236a)$.

ii. $x' = \frac{1}{2}(y - x)$, $y' = \frac{1}{2}(y + x)$; $x = y' - x'$, $y = y' + x'$
require $x'y' = \varepsilon$, $xy = \omega$, and give $N = 16\Omega = 16 N_{vi} \ldots\ldots\ldots (236b)$.

Next, writing, as in Art. 57,

$$N = (x^2 + Cy^2)^2 + \mu y^2 . (Rx^2 + Sy^2) \ldots\ldots\ldots (237),$$

and, taking $\mu = -1, +1, +3, -3$, (as in Art. 57) the coefficients R, S of the type-factorisant (184) become

$$R = (\pm 2C \mp 14). \text{ or } \tfrac{1}{3}(\mp 2C \pm 14); \quad S = (\mp C^2 \pm 1), \text{ or } \tfrac{1}{3}(\pm C^2 \mp 1) \ldots (237a).$$

The procedure is then quite similar to that of Art. 57 *et seq.*, so need not be developed at same length.

78*a*. *Factorisants, Characteristics.* The four Diophantines are styled *Factorisants*, and the quantities (C', C'', C', C^{iv}) on which they depend are styled *Characteristics* for same reasons as in Art. 58, and are said to be of Classes i.–iv. as in that Article.

78*b*. *Equivalence, Reciprocity* (of C_1, C_2). These terms are used in the same sense as in Art. 63. The fundamental conditions are those given in Results (189), (190).

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* When the L, M are composite the partitions are *not unique*, and it is then difficult to select the proper partition of each kind: it is possible that some of the partitions given are not really the proper ones for the purpose.

78*c*. *Equivalence.* In each class the condition (189) leads to $(C_1 - C_2)\{C_1C_2 - 7(C_1 + C_2) + 1\} = 0$.....(238), whence—excluding $C_1 = C_2$—

$$C_2 = (7C_1 - 1)/(C_1 - 7),\quad C_1 = (7C_2 - 1)/(C_2 - 7).....(238a).$$

The possible cases of equivalence will be found to be

$$\text{Classes i., iv.};\quad \frac{C' - 7}{4} = \pm\frac{z'}{z'^0} = \frac{12}{C'^0 - 7}\ .....(238b),$$

$$\text{Classes ii., iii.};\quad \frac{C'' - 7}{4} = \pm\frac{z''}{z'''} = \frac{12}{C''' - 7}\ .....(238c).$$

78*d*. *Reciprocity.* In each class the condition (190) leads to $(C_1C_2 - 2)^2 - (C_2 - 7)^2 - (C_1 - 7)^2 = 101$.....(239).

This condition seems difficult to reduce generally; but, when one of C_1, C_2 is given numerically, the other can be found at once, and (190) with (237*a*) will show to which class it belongs.

78*e*. *Quartetts.* By analogy from Art. 63*c* it may be inferred that the Characteristics exist in sets of four, one in each class, interconvertible by the relations (238*a*, *b*; 239). And the four factorisants arising from them lead to the same Sextan Series, so that any one of them suffices for mere factorisation purposes; usually the factorisant which happens to be easiest of solution would be chosen.

78*f*. 2^{le} *Partitions.* When a solution (x, y, z) of any Factorisant has been obtained, the "2^{le} parts" (P, Q), (a, b), (A, B), (A', B') of the arithmetical partition required for factorisation are given directly by the formulæ (185*a*–*d*); the x, y here belonging of course to the trinomial form (235).

78*g*. *Suitable Characteristics.* This term is used as in Art. 68 to denote Characteristics (C', C'', C''', C^{iv}) which yield *solvable* Factorisants. These—(*i.e.* suitable)—Characteristics may be found by the same process as in Art. 68; and any Sextan (N_0)—prime or composite—may be used as "Base" from which to generate them, and the same formulæ (218*a*), (219*a*), (220*a*), (221*a*) may be used here also [the x_0, y_0 being here of course taken from N_0 expressed in the form (235)].

Factorisants, Examples. The Table below gives a few (nine) selected examples of Characteristics (C') of Class i., and of the Factorisants which they yield, derived from several small Base-Sextans (in the trinomial form),

$$N_0 = 16.13;\quad 16.61;\quad 73;\quad 481 = 13.37;$$

with the data (x_0, y_0, P_0, Q_0) for their formation.

[Note that when $x_0y_0 = \omega$, $N_0 = 16\Omega$ which has two (P, Q) partitions (one with P_0, Q_0 both even, one with them both odd)].

Chain-Examples. The Table below gives six worked Examples from the Factorisants Nos. 1, 3, 5, 6, 7, 9 of the last Table. The x, y, z—(the solutions of the Factorisants)—are of course those of the trinomial form of N: the x', y' of the usual 6-tan form are also given, and the co-factors L, M of N (or of $\frac{1}{16}N$), are arranged so as to exhibit distinctly the chain-property ($M_r = L_{r+1}$).

[When $xy = \omega$, then $N = 16\Omega$, and L, M are *both even*: in this case the even factors (2, 4, 8) of L, M have been divided out, so as to show only the odd co-factors of the Sextan N_{vi}].

TAB. XV.

Factorisants of Class i. of Trinomial Sextans.

$N = x^4 + 14x^2y^2 + y^4$. $\quad (2C'-14)x^2 + (C'^2-1)y^2 = z^2$.

Ex.	N_0	x_0, y_0	P_0 Q_0	C' z_0	*Factorisant*	*Serial*
1	16.13	1, 3	28, 24	3, 8	$-8x^2+8y^2=z^2$	x
2	16.13	3, 1	28, 24	19, 24	$6(2x)^2+10(6y)^2=z^2$	$x, x; y$
3	16.13	1, 3	$\overline{17}$, 9	$\overline{2}$, 3	$-2(3x)^2+3y^2=z^2$	x
4	16.13	3, 1	17, 9	8, 9	$2x^2+7(3y)^2=z^2$	$x, x; y$
5	16.61	1, 5	$\overline{124}$, 120	$\overline{5}$, 24	$-24x^2+24y^2=z^2$	x
6	73	1, 2	37, 36	9, 18	$(2x)^2+5(4y)^2=z^2$	x
7	481	1, 4	241, 240	15, 60	$(4x)^2+14(4y)^2=z^2$	x
8	13.37	1, 4	$\overline{25}$, 12	3/2, 3	$-11x^2+5(\frac{1}{2}y)^2=z^2$	$x; y, y$
9	13.37	4, 1	$\overline{25}$, 12	9, 12	$(2x)^2+5.(4y)^2=z^2$	x, x

Chain Examples from above Factorisants.

Ex. 1. $C'=3$; $y^2-2(\frac{1}{2}z)^2=x^2=+1$; *Ex.* 5. $C'=-5$; $y^2-6(\frac{1}{12}z)^2=x^2=+1$.

x,y,z	1,3,8	1,17,48	1,99,280	1,577,1632	x,y,z	1,5,24	1,49,240	1,485,2376
x', y'	1,2	8,9	49.50	288,289	x', y'	2,3	24.25	242,243
$\frac{1}{4}L$	1;	13;	421;	14281;	$\frac{1}{4}L$	1;	61;	13 457;
$\frac{1}{4}M$	13;	421;	14281;	485113;	$\frac{1}{4}M$	61;	13.457;	37.15733;

Ex. 3. $C'=-2$; $(\frac{1}{3}z)^2-3(\frac{1}{3}y)^2=-2x^2=-2$.

x,y,z	1,3,3	1,9,15	1,33,57	1,123,213	1,459,795	1,1713,2967
x', y'	1,2	4,5	16,17	61,62	229,230	856,857
l	1;	13;	37;	2029;	7057;	13.30241;
m	13;	37;	2029;	7057;	13.30241;	61.22441;

Ex. 6. $C'=9$; $(\frac{1}{2}z)^2-5(2y)^2=x^2=+1$. *Ex.* 7. $C'=15$; $(\frac{1}{4}z)^2-14y^2=x^2=+1$.

x,y,z	1,2,18	1,36,322	1,646,5778		x,y,z	1,4,60	1,120,1796	1.3596,53820
x', y'	1.3	35.37	645.647		x', y'	3.5	119.121	3595.3597
L	1;	73;	13.1789;		L	1;	13.37;	481521;
M	73;	13.1789;	7488433;		M	13.37;	431521;	387504961?†

Ex. 9. $C'=9$; $(\frac{1}{2}z)^2-5(2y)^2=x^2=+16$

Chain 1.	x,y,z	4,1,12	4,21,188	4.377,3372	*Chain* 2.	4,3,28	4,55,492	4,987,8828
	x', y'	3.5	17,25	373,381		1.7	51,59	983,991
	L	13;	37;	7933;		13;	181,	13.4177;
	M	37;	7933;	97.26293;		181;	13.4177;	17480773?†

† No factor < 10000.

79. *Other Diophantine Methods.* Another Method is to attach the characteristic C to the term containing xy, thus—

Let $N = x^4 - x^2y^2 + y^4$ .....(240),

$= (x^2 + Cxy + y^2)^2 - xy.\{2C(x^2+y^2) + (C^2+3)xy\}$.....(240$a$),

$= P^2 - Q^2$ .....(240b),

if $xy.\{2C(x^2+y^2) + (C^2+3)xy\} = Q^2$, a □ .....(240$c$).

This is the factorisant in this Method. It leads to the four algebraically factorisable forms of Art. 38–41, as follows—

1°. Take $C=1$; this leads to $2xy = \square$, [as in Art. 38].

2°. Take $C=3$; this leads to $6xy = \square$, [as in Art. 39].

3°. Take $C = y/x - 2x/y$; this leads to $y^2 - x^2 = 32^2$, [as in Art. 40].

4°. Take $C = -y/x$; this leads to $x^2 - y^2 = z^2$, [as in Art. 41].

Also, with treatment similar to that used in Art. 68, other values of C could be found leading to solvable factorisants. But these factorisants, being of 4th degree in x, y, would be much more difficult of solution than those obtained previously (which were all of 2nd degree): so that it does not seem worth while developing this new Method any further.

80. *Algebraic Factors, (L, M), Diophantine Factorisation.* There is no real difficulty in applying the Diophantine Method above developed for Sextans to the Algebraic Factors (L, M) of the Sextan Forms (B, S', S'', T, D), these factors (L, M) being symmetric* 4-tic forms. It is well to collect here the actual forms of these L-, M-factors from Art. 38, 39, 40, 41.

B has $L, M = \xi^4 \mp 2\xi^3\eta + 2\xi^2\eta^2 \mp 4\xi\eta^3 + 4\eta^4$ .....(241a),

S' has $L, M = \xi^4 \mp 6\xi^3\eta + 18\xi^2\eta^2 \mp 36\xi\eta^3 + 36\eta^4$ .....(241b),

S'' has $L, M = 9\xi^4 \mp 18\xi^3\eta + 18\xi^2\eta^2 \mp 12\xi\eta^3 + 4\eta^4$ .....(241c),

T has $L, M = t^4 \mp 6t^3u + 18t^2u^2 \mp 18tu^3 + 9u^4$ .....(241d),

D has $L, M = t^4 \mp 2t^3u + 2t^2u^2 \mp 2tu^3 + u^4$ .....(241e).

To develop the Method in detail for each of these forms would take too much space. It must suffice to show how the Method is applied to one of them, say to the B-form, and to develop this application in one "class" only, say in Class i., *i.e.* in the $(P^2 - Q^2)$ form of L, M: the application in the other cases being closely similar.

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* This Diophantine Method can be applied with slight modification to the factorisation of any 4-tic form, whatever.

Here the L, M of B may be written, as in Art. 57.

$L,\ M=(\xi^2\mp\xi\eta+C\eta^2)^2-\eta^2.\{(2C-1)\xi^2\mp(2C-4)\xi\eta+(C^2-4)\eta^2\}$

$=P^2-Q^2=P^2-\eta^2z^2$, [upper signs for L, lower signs for M]...(242),

if $(2C-1)\xi^2\mp(2C-4)\xi\eta+(C^2-4)\eta^2=z^2$, (a square).....(242$a$).

This last condition is the *factorisant*. Characteristics (C) which are *suitable—i.e.* which shall yield solvable factorisants—may be found (as in Art. 68) from the values P_0, Q_0 arising from the known value (L_0, M_0) of the L-, M-factor of any Sextan of form B, assumed as Base. The Diophantines thus resulting occur in sets of 4 (as in Art. 63c), and will be all solvable. Every solution (ξ, η, z) thereof will yield the elements ($x=\xi^2$, $y=2\eta^2$) of a new Sextan of form B, along with the factorisation into two factors ($P\mp Q$) of its L, M factor.

[The upper signs in (241a–e) belong to the L-factor, the lower signs to the M-factor: but when ξ has a $-$sign in the solutions of the factorisants, these solutions give the factorisations of the other factor, *i.e.* of the M- or L- factor respectively].

Ex. Take as "Base," $N_0=(1^6+2^6)\div(1^2+2^2)=1.13$;

Here $x_0=1$, $y_0=2$; $\xi_0=1$, $\eta_0=1$; $L_0=1$, $M_0=13=7^2-6^2$.

Hence, taking the M-factor $P_0=\pm7$, $Q_0=6$, $z_0=6$:

And $\xi_0^2+\xi_0\eta_0+C\eta_0^2=\pm P_0$ gives $C=\pm7-2=+5$ or -9.

Taking $C=+5$, the factorisant is $9\xi^2+6\xi\eta+21\eta^2=z^2$,

or $\xi'^2+20\eta^2=z^2$, where $\xi'=3\xi+\eta$.

This has a ξ'-serial solution, (*i.e.* with ξ' constant, Art. 65), but the general solution in terms of two arbitraries (t, u) is more interesting: this is (Art. 64)

$\xi'=t^2-20u^2$, $\eta=2tu$, $z=t^2+20u^2$, $\xi=\frac{1}{3}(\xi'-\eta)$; $x=\xi^2$, $y=2\eta^2$...(242b).

Assigning now various values to t, u (these need not be integers), the following Table is formed; each line is a complete Example.

Ex	*Data* t, u	*Dioph. Solutions* ξ' η ξ z	P Q	*Factors of L or M* $P-Q$, $P+Q$, L, M	N x, y
1	1, $\frac{1}{2}$	4, 1, 1, 6	7, 6	1. 13 M	1, 2
2	2, $\frac{1}{2}$	$\bar{1}$, 2, $\bar{1}$, 9	19, 18	1. 37 L	1, 8
3	3, $\frac{1}{2}$	4, 3, 1/3, 21	415/9, 42	(37/9).(13.61/9), M	1/9, 18
4	8, $\frac{1}{2}$	59, 8, 17, 69	745, 552	193. 1297 M	289, 128
5	1, 1	$\bar{19}$, 2, $\bar{7}$, 21	55, 42	13. 97 L	49, 8
6	7, 1	29, 14, 5, 69	1075, 766	109. 13.157 M	25, 392

The ($P\mp Q$) sub-columns give the factorisation of the L- or M- factor of the Bin-Aurifeuillian Sextan $N=(x, y)$ shown on right.

Ex. 1 is the Base-value (M_0).

Ex. 2, 5 give L-factors as ξ is $-$; *Ex.* 3, 4, 6 give M-factors, as ξ is $+$.

Ex. 3 gives *fractional* values of ξ, P, $P \mp Q$, x; but, multiplying throughout by 9 (to remove fractions), $P=415$, $Q=378$; $M=37.13.61$, the M-factor of $N=(x, y)=(1,162)$.

81. 2^{ic} *Diophantine connexion with Sextans.* From what precedes (Art. 68) it is clear that every Sextan gives rise to a whole group of 2^{ic} Diophantines, and that the Sextan "parts" thereof (x, y) are one solution of the whole group, and lead to an infinity of new solutions (x_r, y_r), each of which gives rise to a factorisable Sextan.

Since, in each derived Sextan (N_r), one of the 2^{ic} "parts" (viz. Q_r, b_r, or a_r, B_r, or B'_r) $=y_r z_r$; and since z_r is usually an integer or fraction at same time as C; it follows that the derived Sextan (N_r) usually has the above 2^{ic} part—(*i.e.* Q_r, &c.) $=y_r$ or $=$ a multiple of y_r, when C is an integer: also, since Cy_r^2 should be an integer (Art. 60), it follows that y_r^2 should contain the denominator of C when C is a fraction.

These conditions limit the nature of the element y_r of all Sextans (N_r) derivable from a given Base (N_0). Hence, although it is certain that every *prime* Base-Sextans (N_0) will generate an infinite number of composite Sextans, it seems possible that composite Sextans are *not* necessarily derivable from prime* Base-Sextans.

82. *Inverse Diophantine Problem.* The question now arises—

"Given two co-factors (L, M) of a *composite* Sextan, to find a Base-Sextan thereof (when such exists)."

Proceed as follows :—

Find the whole set of (secondary) factorisants associated with the given co-factors (L, M) by the process of Art. 72, &c. These factorisants may be all included under the type

$$Rx^2+Sy^2=z^2,$$

and the general problem is from the known solution (x, y, z) of any of the factorisants to find a *lower* solution (x', y', z'), *i.e.* one leading to a lower Sextan (N'), [so that $N'<N$]. This does not appear to be always practicable (even when possible), especially when R, S are fractions: and the conditions for the existence of a lower solution do not appear to be known.

82*a*. *Serial Case.* When the conditions of a Serial Solution (Art. 65) are fulfilled, and when the unit-form

* It is difficult to prove any case of this on account of the number of Diophantines to be solved, and the difficulty of finding the lowest solution of each.

thereof is known, the Problem may be completely tested, and solved (when possible). Proceed thus :—

The formulæ for x_1, y_1, z_1 (201*a*, *b*, *c*) should be used, but *with the −sign*. If these give values

$$y_1,\ z_1 < y,\ z\ ; \quad x_1 = x \qquad \Big|\ \text{or } z_1,\ x_1 > z,\ x\ ; \quad y_1 = y \qquad \Big|\ \text{or } x_1,\ y_1 < x,\ y\ ; \quad z_1 = z$$

then x_1, y_1, z_1 form a lower solution, leading to $N_1 < N$; otherwise a lower (serial) solution of the factorisant tried is impossible (with the unit-form used).

If a lower solution (x_1, y_1, z_1) be found, the same process may be tried again to find a yet lower solution; and so on until the reduction stops: the lowest solution is a Base-form (N_0) which will yield the given Sextan by the direct process.

82*b*. *x-Chain Case*. When the given Sextan (say N_r) is a member of an x-Chain, and the Chain-factors $(L_r,\ M_r)$ thereof are given, the Base-Sextan is easily found. Proceed thus :—

By $$y_{r-1}y_r = L_r \mp x_r^2,\quad y_r y_{r+1} = M_r \mp z_r^2 \ldots\ldots\ldots\ldots (243),$$

whence $$y_{r-1} = \frac{L_r \mp x_r^2}{y_r},\qquad y_{r+1} = \frac{M_r \mp x_r^2}{y_r},\ x_r = x_0 \ldots\ldots\ldots (243a),$$

and the lower solution (x_0, y_{r-1}) has been found.

Again, by (215*b*), $$C' = \frac{y_{r-1} + y_{r+1}}{2y_r} \ldots\ldots\ldots\ldots (243b),$$

Also, by (215*b*), $$y_{r-2} = 2C'.y_{r-1} - y_r \ldots\ldots\ldots\ldots (243c),$$

which gives the next lower values (x_0, y_{r-2}). By repeated use of (243*c*) y_{r-2}, y_{r-3}, &c., may be found in succession; the lowest value of y so obtained is the Base-value (y_0).

Ex. 1°. (of Art. 82*a*). [An x-Series].

Given $\quad N_r = (1^6 + 42^6) \div (1^2 + 42^2) = 673.4621 = L_r.M_r.$

Here $\quad N_r = 2647^2 - 1974^2 = P_r^2 - Q_r^2.$

By (218, 218*a*), $\quad C' = \dfrac{2646}{1764} = \dfrac{3}{2},\quad z_r = \dfrac{1974}{42} = 47.$

The factorisant is $\quad z^2 - \frac{4}{9}y^2 = 4x^4 = 4.1^2.$

The unit-form is $\quad \tau^2 - S.v^2 = (\frac{3}{2})^2 - \frac{5}{9}.1^2 = +1.$

Formulæ (202*b*) with the $-$sign give

$$y_{r-1} = v.z_r \sim \tau.y_r = 1.47 \sim \tfrac{3}{2}.42 = 16\ ;\ [\text{here } y_{r-1} < y_r].$$

The succession-formula $y_{r-2} = 2C'.y_{r-1} - y_r$ gives in succession

$$y_{r-2} = 6,\quad y_{r-3} = 2,\ (\text{the lowest}).$$

Hence $\quad N_0 = (1^6 + 2^6) \div (1^2 + 2^2) = 13$ is the Base-form.

[This example might have been solved at once by "trial" from the factorisant: the lowest solution of this being obviously $(x, y, z) = (1, 2, 3)$].

Ex. 2°. (of Art. 82*b*). [An x- Chain].

Given $N_r = (1^6 + 42^6) \div (1^2 + 42^2) = 673.4621 = L_r . M_r$.

By (243*a*), $\quad y_{r-1} = \dfrac{673-1}{42} = 16, \quad y_{r+1} = \dfrac{4621-1}{42} = 110.$

By (243*b*), $\quad C' = \dfrac{16+110}{2.42} = \frac{3}{2}.$

By (243*c*), $\quad y_{r-2} = 2.\frac{3}{2}.16 - 42 = 6$; and similarly $y_{r-3} = 2$ [as in *Ex.* 1°].

Ex. 3°. [General Case]. Let $N = 1261 = 13.97$;

The complete set of 20 (Secondary) characteristics is given in the Example at end of Art. 73. On comparing these with the set of 20 characteristics of $N_0 = 13$ at end of same Article, it will be seen that four of them are alike, viz.

$$N_0 = 17 \quad \text{has } C_1' = 3/2, \quad C_2'' = 1/2, \quad C_3'' = -6, \quad C_3''' = -5,$$

$$N = 1261 \text{ has } C_1' = 3/2, \quad C_1'' = 1/2, \quad C_7'' = -6, \quad C_3''' \equiv -5,$$

so that they have four factorisants in common. Thus $N_0 = 13$ is evidently a Base-form of $N = 1261$ in four ways, and would have been found from any one of these four (Secondary) factorisants of the latter by the procedure of Art. 82.

83. *Primary and Secondary Characteristics, Relation.* The question arises whether any simple (algebraic) relation exists between the primary and secondary characteristics of a composite Sextan. If any such were discoverable it might lead to a direct factorisation *of given Sextans.*

Example. Taking $N = (1^6 + 6^6) \div (1^2 + 6^2) = 1261 = 13.97$.

The Table at end of Art. 73 gives the complete sets of 20 primary and 20 secondary characteristics. A variety of simple relations between the primaries and secondaries are easily seen: but, on examining the sets of characteristics of other composite Sextans, the same simple relations are not found, so that this is (at present) an *elusive* Problem.

[The discovery of C for a *given* Sextan (N) *i.e.* such as to give the expression of N in form $(P^2 - Q^2)$, &c., involves in fact a prior knowledge of P, Q, &c.]

Chapter V. *Residuacity.*

[This short Chapter deals with the residuacity of small bases (q) towards Sextan primes].

84. *Residuacity, Modularity.* The relation of a number q to a prime (p)—depending on Fermat's theorem—may be such that

$$q^{\frac{1}{n}(p-1)} \equiv \pm 1 \pmod{p}; \quad [n \text{ a factor of } (p-1)] \ldots\ldots (244),$$

which is written shortly as

$$(q/p)_n = \pm 1.$$

In this connection q is called the *Base*, and p the *Modulus*: and—[when $(q/p)_n = +1$]—q is said to be a n-ic *Residue* of p, and p is said to be a n-ic *Modulus* of q. The property (244) is called the *Residuacity* of q, or the *Modularity* of p.

The criteria of n-ic residuacity of $+q$ and $-q$ are connected by the relations—

$(q/p)_n = (\overline{q}/p)_n$, when $n = \omega$;

$$(q/p)_n \cdot (\overline{q}/p)_n = (\overline{1})^{\frac{1}{n}(p-1)}, \text{ when } n = \epsilon \ldots (245).$$

85. *Modularity of 6-tans and 12-mans.* The criteria of the 2^{ic}, 4^{ic}, 8^{ic}, and 3^{ic} modularity of 6-tan and 12-man *primes* towards many bases $(\pm q)$ can be readily expressed in terms of the 6-tan and 12-man elements (x, y) when the criteria for such bases $(\pm q)$ are known in terms* of the (a, b), (c, d), (A, B), (L, M) partitions of the moduli (p), because these partitions are known (algebraically) in the case of 6-tan and 12-man primes (Art. 3, 4).

86. *Tables of Modularity of 6-tan primes.* Tables (XVI. to XVIII.) at the end of this Paper give the criteria of n^{ic} residuacity of certain bases $(\pm q)$ towards 6-tan primes $(p = N_{vi})$† only.

Tab.	*Residy.*	*Bases* $(\pm q)$.
XVI.	$n = 2, 4$	2, 3, 5, 6, 7, 10, 11, 12.
XVII.	$n = 8$	2, 3, 6, 12.
XVIII.	$n = 3$	2, 3, 5, 6, 7, 10, 11, 12.

[Note that the 8^{ic} modularity of 6-tan primes admits of *algebraic* determination only for the bases $q = \pm 2$, ± 3, $\pm 2^{\alpha}.3^{\beta}$; because the (a, b) partition suffices for the case of $y = \pm 2$, and the (a, b), (A, B) partitions together suffice‡ for the case of $q = \pm 3$].

* In the cases of the bases $\pm q = 2, 3, 5, 6, 7, 10, 11, 12$, these criteria are given in a convenient form in the late Mr. C. E. Bickmore's, two Papers *On the numerical factors of* $(a^n - 1)$ in the *Messenger of Mathematics*, Vols. xxv. & xxvi., 1905-6. Some Errata therein require correction. Extensive Tables of such criteria for all bases $\pm q \geqslant 50$ have been prepared by the present author: it is hoped to publish these shortly.

† The number of 12-man primes known is so few that it has not been thought worth while drawing up similar Tables for them.

‡ This criterion for $(\pm 3/p)_8$ in terms of the (a, b), (A, B) partitions was discovered by the author by induction from a large number of numerical cases. It was included in a Paper by the present author, read before the London Mathematical Society in 1909, but not yet published: it has since been *proved* by Mr. A. E. Western, but the proof has not yet been published. The application to 6-tan moduli follows immediately.

87. *New Criterion for* $(3/p)_8$. The criterion for $(3/p)_8$ in terms of (a, b), (A, B)—alluded to above—is as follows. The preliminary conditions

$$p = 8\varpi + 1 \text{ and } (3/p)_4 = +1 \quad \ldots\ldots\ldots\ldots (246),$$

involve

$$p = a^2 + b^2 = A^2 + 3B^2 \quad \ldots\ldots\ldots\ldots (246a),$$

where $a = 6\alpha \pm 1$, $b = 12\beta$, $A = 12\alpha \pm 1$, $B = 4b \ldots (246b)$,

Then the criterion is

$$(3/p)_8 = (\bar{1})^{\mu}, \text{ where } \mu = \alpha + b \quad \ldots\ldots\ldots\ldots (247),$$

so that $(3/p)_8 = -1$, if $\mu = \omega$; $(3/p)_8 = +1$, if $\mu = \varepsilon \ldots (247a)$.

TAB. XVI.

Criteria of $(q/p)_2 = \mp 1$ *and* $(q/p)_4 = \mp 1$.

$\pm q = 2, 3, 5, 6, 7, 10, 11, 12$; $p = N_{ui} = (x^6 + y^6)/(x^2 + y^2)$; $Z = xy(x \pm y)$.

q	$x,\ y$	$p-1$	$(q/p)_2=-1$	$(q/p)_2=+1$	$(q/p)_4=-1$	$(q/p)_4=+1$
± 2	$y=\varepsilon$	4ϖ	$y=2\omega$	$y=4\eta$	$y=4\omega$	$y=4\varepsilon$
± 2	$xy=\omega$	8ϖ	.	$xy=\omega$	.	$xy=\omega$
$+3$	$y=2\omega$	4ω	.	$y=2\omega$	$xy=6\omega$	$xy\mp 6\omega$
-3	$y=2\omega$	4ω	.	$y=2\omega$	$xy\mp 6\omega$	$xy=6\omega$
± 3	$y=4\eta$	16ϖ	.	$y=4\eta$	$xy\mp 6\varepsilon$	$xy=6\varepsilon$
± 3	$xy=\omega$	8ϖ	.	$xy=\omega$	$xy=3\omega$	$xy\mp 3\omega$
$+5$	$y=2\omega$	4ω	$Z\mp 10\omega$	$Z=10\omega$	$xy\mp 10\omega$	$xy=10\omega$
-5	$y=2\omega$	4ω	$Z\mp 10\omega$	$Z=10\omega$	$xy=10\omega$	$xy\mp 10\omega$
± 5	$y=4\eta$	16ϖ	$Z\mp 20i$	$Z=20i$	$xy\mp 10\varepsilon$	$xy=10\varepsilon$
± 5	$xy=\omega$	8ϖ	$Z\mp 40i$	$Z=40i$	$xy=5\omega$	$x\mp y=5\varepsilon$
± 6	$y=\varepsilon$	4ϖ	$y=2\omega$	$y=4\eta$	$xy=12\omega$ $xy=4\varepsilon+3i$	$xy=12\varepsilon$ $xy=4\omega+3i$
± 6	$xy=\omega$	8ϖ	.	$xy=\omega$	$xy=3\omega$	$xy\mp 3\omega$
$+7$	$y=2\omega$	4ω	$Z\mp 14\omega$	$Z=14\omega$	$Z=14\omega$	.
-7	$y=2\omega$	4ω	$Z\mp 14\omega$	$Z=14\omega$	.	$Z=14\omega$
± 7	$y=4\eta$	16ϖ	$Z\mp 28i$	$Z=28i$	.	$Z=14\varepsilon$
± 7	$xy=\omega$	8ϖ	$Z\mp 28i$	$Z=28i$	.	$Z=14\varepsilon$
$+10$	$y=2\omega$	4ω	$Z\mp 10\omega$	$Z=10\omega$	$2x\pm y=20\varepsilon$	$2x\pm y=20\omega$
-10	$y=2\omega$	4ω	$Z\mp 10\omega$	$Z=10\omega$	$2x\pm y=20\omega$	$2x\pm y=20\varepsilon$
± 10	$y=4\eta$	16ϖ	$Z\mp 20i$	$Z=20i$	$xy=20\omega$ $x\pm y=5\omega,\ y=4\varepsilon$	$xy=20\varepsilon$ $x\pm y=5\omega,\ y=4\omega$
± 10	$xy=\omega$	8ϖ	$Z\mp 10i$	$Z=10i$	$xy=5\omega$	$x\pm y=10\eta$
$+11$	$y=2\omega$	4ω	$Z\mp 22\omega$	$Z=22\omega$	$xy=22\omega$	$x\pm y=11\omega$
-11	$y=2\omega$	4ω	$Z\mp 22\omega$	$Z=22\omega$	$x\pm y=11\omega$	$xy=22\omega$
± 11	$y=4\eta$	16ϖ	$Z\mp 44i$	$Z=44i$	$x\pm y=11\omega$	$xy=44i$
± 11	$xy=\omega$	8ϖ	$Z\mp 22i$	$Z=22i$	$xy=11\omega$	$x\pm y=22i$
$+12$	$y=2\omega$	4ω	.	$y=2\omega$	$xy\mp 6\omega$	$xy=6\omega$
-12	$y=2\omega$	4ω	.	$y=2\omega$	$xy=6\omega$	$xy\mp 6\omega$
± 12	$y=4\eta$	16ϖ	.	$y=4\eta$	$xy\mp 6\varepsilon$	$xy=6\varepsilon$
± 12	$xy=\omega$	8ϖ	.	$xy=\omega$	$xy=3\omega$	$xy\mp 3\omega$

TAB. XVII.

Criteria of $(q/p)_8 = \mp 1$.

$\pm q = 2,\ 3,\ 6,\ 12\ ;\qquad p = N_{vi} = (x^6 + y^6)/(x^2 + y^2).$

q	$x,\ y$	$p-1$	$(q/p)_8 = -1$	$(q/p)_8 = +1$
± 2	$y = 8\eta$	16ϖ	$y = 8\omega$	$y = 8\varepsilon$
$+2$	$xy = \omega$	8ϖ	.	$x \pm y = 4\omega$
-2	$xy = \omega$	8ϖ	$x \pm y = 4\omega$	.
± 2	$xy = \omega$	16ϖ	.	$x \pm y = 8\eta$
± 3	$y = 4\eta$	16ϖ	$x = 3\omega$	$y = 12\eta$
$+3$	$xy = \omega$	8ω	$x \pm y = 12\omega$ & 2ω	$x \pm y = 6\omega$ & 4ω
-3	$xy = \omega$	8ω	$x \pm y = 6\omega$ & 4ω	$x \pm y = 12\omega$ & 2ω
± 3	$xy = \omega$	16ϖ	$x \pm y = 6\omega$ & 8η	$x \pm y = 12\varepsilon$ & 2ω
± 6	$y = 8\eta$	16ϖ	$y = 24\omega\ ;\ x = 3\omega$ & $y = 8\varepsilon$	$xy \mp 3i$, &c.*
± 6	$y = 4\omega$	16ϖ	$xy \mp 3i$, &c.*	$y = 24\varepsilon\ ;\ x = 3\omega$ & $y = 8\omega$
$+6$	$xy = \omega$	8ϖ	$x \pm y = 2\omega$ & 12ω	$x \pm y = 6\omega$ & 4ω
-6	$xy = \omega$	8ϖ	$x \pm y = 6\omega$ & 4ω	$x \pm y = 2\omega$ & 12ω
± 6	$xy = \omega$	16ϖ	$x + y = 6\omega$ & 8η	$x \pm y = 2\omega$ & 12ε
± 12	$y = 4\eta$	16ϖ	$y = 12\omega\ ;\ x = 3\omega$ & $y = 4\varepsilon$	$y = 12\varepsilon\ ;\ x = 3\omega$ & $y = 4\omega$
$+12$	$xy = \omega$	8ω	$x \pm y = 6\varepsilon$	$x \pm y = 6\omega$
-12	$xy = \omega$	8ω	$x \pm y = 6\omega$	$x \pm y = 6\varepsilon$
± 12	$xy = \omega$	16ϖ	$x + y = 6\omega$	$x \pm y = 6\varepsilon$

* The condition here required is not easily expressed in terms of $x,\ y$.

TAB. XVIII.

Criteria of $(q/p)_3 = 1$.

$q = 2,\ 3,\ 5,\ 6,\ 7,\ 10,\ 11,\ 12\ ;\qquad p = N_{vi} = (x^6 + y^6)/(x^2 + y^2).$

q	*Prelim^y.*	*Criterion*	*Prelim^y.*	*Criterion*
2	$xy \mp 3i\ ;$	$xy = \omega$	$xy = 3\iota\ ;$	$x = 6\xi,$ or $y = 6\eta$
3	$xy \mp 3i\ ;$	$x \pm y = 9i$	$xy = 3i\ ;$	$x = 3\xi,$ or $y = 3\eta$
5	$xy \mp 3i\ ;$	$xy \mp 5i$	$xy = 3i\ ;$	$x = 15\xi,$ or $y = 15\eta$
6	$xy \mp 3i\ ;$	$\left\{ \begin{array}{l} xy = \omega \ \&\ x \pm y = 18i \\ x = \omega,\ y = \varepsilon,\ x \pm 2y = 9\omega \end{array} \right\}$	$xy = 3i\ ;$	$x = 6\zeta,$ or $y = 6\eta$
7	$xy \mp 3i\ ;$	$x \pm y = 7i$	$xy = 3i\ ;$	$\left\{ \begin{array}{l} x = 3\xi \ \&\ x(y \pm 2x) = 7i \\ y = 3\eta \ \&\ y(x \pm 2y) = 7i \end{array} \right.$
10	$xy = \omega\ ;$	$xy \mp 5i$	$xy = \varepsilon\ ;$	$x = 10\xi,$ or $y = 10\eta$
11	$xy \mp 3i\ ;$	$x \pm y = 11i$	$xy = 3i\ ;$	$\left\{ \begin{array}{l} x = 3\xi \ \&\ x(x \pm 4y)(x \pm 4y) = 11i \\ y = 3\eta \ \&\ y(y \pm 2x)(y \pm 4x) = 11i \end{array} \right.$
12	$xy \mp 3i\ ;$	$\left\{ \begin{array}{l} xy = \omega \ \&\ x \pm y = 18i \\ x = \omega,\ y = \varepsilon\ (2x \pm y) = 18i \end{array} \right\}$	$xy = 3i\ ;$	$x = 6\xi,$ or $y = 6\eta$

Addenda and Corrigenda in Chap. I.—III. Vol. xxxix.

Addenda to Tab. I. (page 59).

p	x	y	p	x	y	p	x	y
5899273	51,	47	8047801	55,	51	8543881	55,	53
7349473	53,	51	8288641	57,	47	8765101	47,	58
7378081	39,	56	8265121	25,	56	8832721	19,	56
7652401	55,	49	8487373	41,	58	8839021	33,	58
						8915953	53,	56

Corrigenda.

Page 35. In Result (8*d*); *For* $+\delta\phi$, *Read* $=\delta\phi$.
,, 35. In Result (8*e*); *For* $-\frac{1}{2}(e+c)$, *Read* $=\frac{1}{2}(e+c)$.
,, 37. In Result (16*a*); *For* $2(x+y)^2$, *Read* $3(x^2\sim y^2)^2$.
,, 38. In Result (18); *For* N_{iv}, *Read* N_{vi}.
,, 39, l. 15; *For* divisors, *Read* divisor.
,, 40, l. 4; *For* $+2.1044^2$, *Read* -2.1044^2.
,, 40, l. 26; *For* 332, *Read* 345.
,, 46. In Result (31); *For* 2*A*, *Read* 2*B*.
,, 52. In Result (51*a*); *For* $\frac{1}{2}(y'+y')$, *Read* $\frac{1}{2}(y'+x')$.
,, 59, Tab. I.; *For* $p=4854761$, *Read* 4854781.
,, 60, Tab. II.; *For* $p=2847242961$, *Read* 2847342961.
,, 61, Tab. III.; *For* $p=9122481$, *Read* 10122481.
,, 61, Tab. III.; *For* $p=17764857$, *Read* 17864857.
,, 62, Tab. III.; *For* $p=438517391$. *Read* 438517393.
,, 109. In upper Table; *For* x, $y=99,101$, *Read* 101,99.
,, 114, l. 4 up; *For* (136), *Read* (146).
,, 115. In lower Table; Headline: *For* t'', u'', *Read* t, u.
,, 116. In Result (148*b*); *For* η_{r-1}, *Read* η_{2^r-1}.
,, 128. *For* Art. 89, *Read* Art. 39.

Corrigenda on the author's Paper on High Quartan Factorisations, &c, in Vol. xxxvi.

Page 145, Art. 3. Correct as follows:—

Condition	*Half-Quartans*	*Half-Octavans*
x or $y=3i$	*Read* $8.3n+17$	*Read* $16.3n+17$
x or $y=5i$	*Cancel* $8.5n+1$	*Cancel* $16.5n+1$

Page 146, In Result (6), *Cancel* 1′.
,, 147, l. 20; *For* "an *odd*," *Read* "a *prime*."
,, 147, In Result (4*c*′); *For*), *Read*].
,, 147; Add at end of Art. 4*a*:—

When N, or $\frac{1}{2}N$, is *composite*, it will be expressible as a Quartan, or Half-Quartan, if *some one* of its 2^{4c} forms of *each kind* (3*a*—*d*, 4*a*—*d*) satisfies the appropriate condition (3*a*′—*d*′, 4*a*′—*d*′).

Page 165; *For* Results (41*b*), (41*b*), *Read* (41*a*), (41*b*).
,, 167; *For* Art. (32), *Read* Art. (23).
,, 169, Tab. I. *For* $p=8124461$, *Read* 8124161.

A NOTE ON A CLASS OF SYMMETRIC FUNCTIONS AND ON A THEOREM REQUIRED IN THE THEORY OF INTEGRAL EQUATIONS.

By *W. H. Young, Sc.D., F.R.S.*

§ 1. In a recent paper in the *Messenger of Mathematics*,* Mr. Bateman has considered certain questions connected with the Integral Equation of the First Kind. His treatment of these questions is conditioned by the fact that a certain theorem, which he tells us is due to Lerch, is stated only for continuous functions. In § 2 of the present Note I show how, by means of a method recently described in a paper to appear in the *Proceedings of the London Mathematical Society*,† the corresponding theorem may be proved for any kind of function, bounded or unbounded, provided only that they belong to the class of summable functions. In the case of bounded functions this involves no restriction, except one of a philosophical, rather than of a mathematical, nature, and the only further restriction, if the function is unbounded, is that we exclude integration of a type other than that known as absolute.‡ It will be seen that the proof here given is an excellent example of the facility with which the method referred to may be employed in practice.

The object of §§ 6—9 of the Note is to show how the same method gives us an intuitive proof of the fact that *the necessary and sufficient conditions that*

$$\int_a^b \int_a^b k(s, t)\, \theta(s)\, \theta(t)\, ds\, dt$$

should be positive or zero, when $k(s,t)$ is an assigned symmetric and continuous function and θ runs through the whole class of

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* H. Bateman, "Notes on Integral Equations," *Mess. of Math*, vol. XXXIX., pp. 129—135.

† W. H. Young, "On a New Method in the Theory of Integration," *Proc. L.M.S.*, 1910.

‡ It may also be noted for the extension of the results in question, we require in addition the theorem that if $f(s, t)$ is a bounded function and $g(t)$ is any summable function, change of order of integration in the integral

$$\int_0^a \int_0^b g(t) f(s, t)\, ds\, dt$$

is always allowable. In fact, this is obviously true when $f(s, t)$ is positive, by de la Vallée Pouissin's Theorem, and therefore in the general case. The reader may consult a recent paper by the author in a forthcoming number of the *Transactions of the Cambridge Philosophical Society*, in which the whole question of change of order of integration in a repeated integral is discussed.

summable functions, are identical with the conditions that the quadratic form in n variables

$$\Sigma\{k(a_i, a_j)x_i^2 + 2k(a_i, a_j)x_i x_j\} \quad (i, j = 1, 2, \ldots, n),$$

*where $a_1, a_2, \ldots, a_n$ are any n points of the interval (a, b), should be positive or zero, whatever the number n of points chosen, and however these points be selected.**

§ 2. THEOREM. *If $h(s)$ is any bounded function other than a null-function (where, as usual, we assume tacitly that the function is also measurable in Lebesgue's sense), then*

$$\int_a^b s^n h(s)\, ds$$

cannot be zero for all positive integral values of n including zero.

For, if the contrary were the case, the integral would vanish when we substitute for s^n any polynomial. Now, by Weierstrass's Theorem, any continuous function can be expressed as the limit of an uniformly converging series of polynomials. Such a series, when multiplied by the bounded function $h(s)$, is, of course, by Lebesgue's Theorem, capable of term-by-term integration. It would accordingly follow that the integral would vanish when we substituted for s^n any continuous function whatever.

But again a lower semi-continuous function which is bounded is expressible as the limit of a monotone ascending sequence of continuous functions, and, as the points of non-uniform convergence are completely visible, such a sequence has no infinite peaks or chasms of non-uniform convergence. Thus we may again apply Lebesgue's Theorem to such a sequence when multiplied by $h(s)$.

* In Hilbert's treatment of Fredholm's Theory of Integral Equations, the class of symmetric functions $k(s, t)$, for which the double integral

$$\iint k(s, t)\, \theta(s)\, \theta(t)\, ds\, dt$$

is greater than zero, plays a very important part. Hilbert's discussion is based on the analogy between the double integral and the quadratic form in any number of variables. For Hilbert's purpose the limitation "greater than zero" is essential and cannot be replaced by "greater than or equal to zero."

On the other hand, the discussion of the *properties* of the class of functions, for which the restriction is "greater or equal to zero," will naturally be simpler than that of the class utilised by Hilbert. This follows from the fact that the limit of a sequence of functions greater than zero is not necessarily greater than zero, whereas the limit of a sequence of functions ≥ 0 is ≥ 0. The character of the reasoning employed in the present note sufficiently brings out the connection with the Theory of Limits. The proof of the necessity and sufficiency of the conditions in question is one of the main objects of a memoir by Mr. Mercer, recently communicated to the Royal Society by Dr. Forsyth, and printed in the *Phil. Trans.*, series A, vol 209, pp. 415–446. Our result is indeed more general than the corresponding one in Mr. Mercer's memoir, as in that memoir θ is confined to the class of continuous functions.

Then

$$\int_a^b B(s)\, h(s)\, ds = 0$$

for every bounded lower semi-continuous function $B(s)$.

Similarly $$\int_a^b A(s)\, h(s)\, ds = 0$$

for every bounded upper semi-continuous function $A(s)$.

Similarly, by precisely the same argument, these equations hold if B is any *ul*-function or A any *lu*-function.

Now, given any bounded function, we can find a bounded *ul*-function and a bounded *lu*-function differing only from it at a set of content zero,* these are called *bounding ul- and lu-functions of the given function.* Hence the above holds whatever bounded functions $A(s)$ and $B(s)$ may be. Hence, choosing, for example, $h(s)$ itself for $A(s)$,

$$\int_a^b \{h(s)\}^2 ds = 0,$$

which is only possible if $h(s)$ is zero, except at a set of content zero. Since, by hypothesis, this is not the case the theorem follows by a *reductio ad absurdum.*

§ 3. We have purposely written out the preceding proof for the case of a bounded function $h(s)$. The theorem, however, remains true whether $h(s)$ be bounded or not. In other words we may assert that, if $\int_a^b s^n h(s)\, ds = 0$, for all positive integral values of n, including zero, then $h(s)$ is necessarily a bounded or unbounded null-function, that is to say it is zero at all points excepting a set of content zero.

In order to prove this extended statement we need certain theorems in the theory of integration of sequences of unbounded functions, which will be found in the following article, viz., Lemmas 1 and 2. Assuming these lemmas, we notice that the argument already used applies, as it stands, to show that

$$\int_a^b f(s)\, h(s)\, ds = 0,$$

where $f(s)$ is any bounded (measurable) function whatever.

* An *ul*-function is the limit of a monotone descending sequence of lower semi-continuous functions; a *lu*-function is the limit of a monotone ascending sequence of upper semi-continuous functions. See my paper quoted above. The theorem used follows also from the fact that a bounded function agrees with the derivates of its integral except at a set of content zero.

Take then as $f(s)$ the function which is positive unity wherever $h(s)$ is positive, and negative unity wherever $h(s)$ is negative. This gives

$$\int_a^b |h(s)|\, ds = 0,$$

which is only possible if the points at which $|h(s)|$ is positive, not zero, form a set of content zero. Thus $h(s)$ is a null-function contrary to hypothesis.

§ 4. It is evident that we may deduce at once that the theorem is equally true, if, instead of requiring the vanishing of the integral for every positive value of n, including zero, we only require n to be positive, excluding zero, or only require n to be even, or to be odd, and the like. Such a hypothesis will merely require us to make a preliminary transformation of $h(s)$, and, if necessary, of the independent variable. It would be interesting to obtain some indications as to the law of choice of indices allowable.

§ 5. The following are the lemmas alluded to in the preceding article:—

Lemma 1. *If $h(s)$ is a summable function (not necessarily bounded either above or below), and if $f_1(s)$, $f_2(s)$, ... be a sequence of continuous functions, converging uniformly to a limiting function $f(s)$,*

$$\operatorname*{Lt}_{n=\infty} \int_a^b f_n(s)\, h(s)\, ds = \int_a^b f(s)\, h(s)\, ds.$$

For, supposing in the first instance that $h(s)$ is essentially positive, *i.e.*, ≥ 0, we are given that $f_n(s)$ is a continuous function of (n, s) at the point (∞, s), hence the integral on the left is a continuous function of n at the point $n = \infty$.*

But, by the definition of a summable function, $h(s)$ may be split up into the difference $h_1(s) - h_2(s)$ of two essentially positive summable functions.

Since the theorem is true of each of these two functions separately, it is true of their difference, which proves the theorem.

Lemma 2. *If $h(s)$ is, as before, a summable unbounded function, and if $f_1(s)$, $f_2(s)$, ... be a monotone sequence of bounded functions, converging to a bounded limiting function $f(s)$, the same is true.*

* The theorem here used is a particular case of Theorem 5, p. 143 of my paper, "On Parametric Integration," *Monatshefte für Math. u. Physik* (1910).

Suppose, for definiteness, the sequence is monotone increasing. Then it is sufficient to prove the theorem for a positive function $h(s)$. But, in this case, the sequence $h(s)f_1(s)$, $h(s)f_2(s)$, ... is also monotone ascending, and, since $f(s)$ is bounded, the limiting function of this new sequence is summable. Thus the result stated follows by the method of monotone sequences.

Lemma 3. *The result of Lemma* 2 *remains true if* $f_1(s)$, $f_2(s)$, ... *be unbounded, and converge to an unbounded limiting function, provided only the various integrals which occur exist.*

The Method of Monotone Sequences, in fact, applies equally in this case.

§ 6. We turn now to the double integral

$$\int_a^b\int_a^b k(s,\,t)\,\theta(s)\,\theta(t)\,ds\,dt$$

and first show that *if the double integral in question is* ≥ 0 *for all continuous functions* θ, *it is so for all bounded functions.*

Since a lower semi-continuous function is the limit of a monotone ascending sequence of continuous functions, we have, replacing $\theta(s)$ by one of its bounding *ul*-functions $b(s)$, and denoting the lower semi-continuous functions of which $b(s)$ is the limit by b_n, and the continuous functions of which b_n is the limit by $b_{n,m}$,

$$\int_a^b\int_a^b k(s,\,t)\,\theta(s)\,\theta(t)\,ds\,dt = \int_a^b\int_a^b k(s,\,t)\,b(s)\,b(t)\,ds\,dt,$$

since $\theta(s)$ and $b(s)$, $\theta(t)$ and $b(t)$ differ only at a set of content zero. This is again

$$= \operatorname*{Lt}_{n=\infty} \int_a^b\int_a^b k(s,\,t)\,b_n(s)\,b_n(t)\,ds\,dt$$

$$= \operatorname*{Lt}_{n=\infty}\operatorname*{Lt}_{m=\infty} \int_a^b\int_a^b k(s,\,t)\,b_{n,m}(s)\,b_{n,m}(t)\,ds\,dt,$$

since the functions involved are all bounded, so that the integral of the limit is the limit of the integral. The last integral is, by hypothesis, ≥ 0, therefore the double integral in question is also ≥ 0.

§ 7. We can even go a step farther and remove the restriction that $\theta(s)$ should be bounded; $\theta(s)$ *may, in fact, belong to the class of unbounded summable functions in the sense*

of Lebesgue. Not only can we assert that $\int_a^b k(s, t)\,\theta(s)\,ds$ is a bounded function of t, as it lies between $A\int_a^b \theta(s)\,ds$ and $B\int_a^b \theta(s)\,ds$, where A and B are the upper and lower bounds of $k(s, t)$, we know further that it is a continuous function of t. Thus $\int_a^b\int_a^b k(s, t)\,\theta(s)\,\theta(t)\,ds\,dt$ will exist. Moreover,

$$\theta(s) = \operatorname*{Lt}_{n=\infty}\ \operatorname*{Lt}_{m=\infty}\ \theta_{m,n}(s),$$

where $\theta_{m,n}(s)$ has the value $\theta(s)$, wherever $-m \leq \theta(s) \leq n$, and is zero elsewhere.

Considering then first integration with respect to s, and then with respect to t, and bearing in mind that term-by-term integration of a monotone sequence, which is itself integrable, remains allowable when each function of the sequence is multiplied by any the same bounded function, the required result follows in the same manner as in the preceding article.

The extension of the present article is not essential for our purpose. It should be noted, however, that the substitution of *bounded* for *continuous* functions is not without importance, even were it demonstrable that the only case of interest were that in which $\theta(s)$ belongs to the class of continuous functions. It is easier, in fact, to word the proof of the necessity of the conditions in the more general than in the simpler case.

§ 8. That *the conditions mentioned in* § 1 *are sufficient* to make the double integral in question ≥ 0 is at once suggested by the expression for the double integral as a single limit, in the case in which $\theta(s)$ is a continuous function, viz.,

$$(b-a)^2 \operatorname*{Lt}_{n=\infty}\ n^{-2}\Sigma\left[k(a_i, a_i)\,\theta(a_i)\,\theta(a_i) + 2k(a_i, a_j)\,\theta(a_i)\,\theta(a_j)\right],$$

where $a_1, a_2, \ldots, a_n$ are points by means of which (a, b) is divided into $(n+1)$ equal parts.

Assuming then that the conditions hold, this is the limit of an expression which is ≥ 0 for all values of n and all choices of the a's; hence the limit is also ≥ 0, when $\theta(s)$ belongs to the class of continuous functions, and therefore, by what proceeds, when $\theta(s)$ is any bounded function whatever.

§ 9. We have next to prove that *the conditions are necessary.*

For this purpose we take any n points on the diagonal of the square, which is the field of integration,

$$(a_1, a_1),\ (a_2, a_2),\ \ldots,\ (a_n, a_n),$$

and complete this system by adding to it all the points (a_i, a_j) for all the integers i, j from 1 to n inclusive. We can determine a length h such that, if squares of side h be described round the n^2 points (a_i, a_j) as centres, the oscillation of $k(s, t)$ inside each is less than e, where e is a chosen positive quantity, as small as we please.

Choose a function $\theta(s)$ as follows:—it is to be zero for every value of s which is not the abscissa of a point inside one of these squares. For a value of s corresponding to the abscissa of a point inside the square whose centre is (a_i, a_j), we have

$$\theta(s) = x_i/h,$$

where x_i is a selected constant. Thus the value of the product $\theta(s)\,\theta(t)$ at a point inside this square is $x_i x_j/h^2$. Therefore the double integral, taken over this square, is

$$x_i x_j \int k(s, t)\, ds\, dt/h^2 \leq x_i x_j k(a_i, a_j) + e\,|x_i x_j|.$$

Hence the whole double integral is not greater than

$$\Sigma\,[x_i^2 k(a_i, a_j) + 2x_i x_j k(a_i, a_j)] + e\,[|x_1| + |x_2| + \ldots + |x_n|]^2.$$

Since, by hypothesis, the double integral ≥ 0, the expression just found ≥ 0.

Keeping n fixed, and letting e diminish down to its limit zero, we get

$$\Sigma\,\{x_i^2 k(a_i, a_i) + 2x_i x_j k(a_i, a_j)\} \geq 0.$$

Since this is true for all values of n, it proves that the conditions are necessary if the double integral is to be > 0 for discontinuous functions $\theta(s)$.

It at once follows that the condition is necessary if the double integral is to be > 0 for continuous functions $\theta(s)$. For, if the double integral > 0 for all continuous functions $\theta(s)$, it is so for the set of functions having as limit, or repeated limit respectively, a semi-continuous function, or an *ul*-function, and therefore, by the method of sequences, it is so for every kind of bounded discontinuous function. Hence the conditions just obtained must hold in this case also; they are necessary. Thus the necessity and the sufficiency of the conditions are established.

Geneva,
June 17th, 1910.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

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XXVII.

Oscillating cases of Dirichlet's integral (continued).

§ 1. In two previous notes*, bearing the same titles as this one, I discussed the behaviour, as $\lambda \to \infty$, of the integral

$$(1) \qquad \int_0^\xi F(x)\,\phi(1/x) \begin{matrix}\cos\\ \sin\end{matrix} \{\psi(1/x)\}\,\frac{\sin \lambda x}{x}\,dx,$$

where $\phi(1/x)$ and $\psi(1/x)$ are functions of the logarithmic scale, the latter tending to ∞ not more rapidly than $\log(1/x)$, and $F(x)$ being regular in every way at the origin. The general result of the discussion was that the integral tends to zero if and only if $\phi(1/x)$ does so, and otherwise oscillates.

When $\psi(1/x)$ is higher than $\log(1/x)$ in the logarithmic scale the discussion is considerably more difficult. The case which first suggests itself is naturally that in which

$$\psi(1/x) \sim 1/x.$$

And I propose to begin by considering the particular integral for which

$$F(x) = 1, \quad \phi(1/x) = 1/x^t, \quad \psi(1/x) = m/x,$$

where $m > 0$, and the sine of $\psi(1/x)$ is chosen. This integral is

$$\int_0^\xi \sin\left(\frac{m}{x}\right) \sin \lambda x\, \frac{dx}{x^{1+t}}.$$

It is convergent if $t < 2$, and remains convergent, when the upper limit is replaced by ∞, if $t > -2$. And it is easy to see (cf. Note XXIII., § 9) that in this latter case

$$\int_\xi^\infty \sin\left(\frac{m}{x}\right) \sin \lambda x\, \frac{dx}{x^{1+t}} \to 0$$

as $\lambda \to \infty$; so that replacing ξ by ∞ does not affect the problem which we are considering. Moreover, if we put

$$\lambda x = u, \quad m\lambda = b, \quad 1 + t = s,$$

* Note XXIII., vol. XXXVIII., p. 1; and Note XXIV., vol. XXXVIII., p. 176. For some corrections to Note XXIV., see the end of this note.

we obtain

$$\int_0^\infty \sin\left(\frac{m}{x}\right)\sin\lambda x\,\frac{dx}{x^{1+t}} = \lambda^{s-1}\int_0^\infty \sin u \sin\left(\frac{b}{u}\right)\frac{du}{u^s},$$

where $b>0$ and $-1<s<3$.

§ 2. The last integral is one of some interest in itself, and I shall now proceed to evaluate it. Its form suggests that it should be capable of expression in terms of Bessel Functions.

Let

$$(2)\quad I(b)=\int_0^\infty \sin u \sin\left(\frac{b}{u}\right)\frac{du}{u^s},\quad J(b)=\int_0^\infty \cos u\cos\left(\frac{b}{u}\right)\frac{du}{u^s},$$

where $b>0$. We shall suppose in the first instance that $0<s<1$. Then

$$\frac{dI}{db} = \int_0^\infty \sin u\cos\left(\frac{b}{u}\right)\frac{du}{u^{s+1}},$$

$$\frac{d^2I}{db^2} = -\int_0^\infty \sin u\sin\left(\frac{b}{u}\right)\frac{du}{u^{s+2}},$$

it being easy to prove that the last two integrals are uniformly convergent with respect to b in any interval $0<b_0\leq b\leq b_1$.* Integrating the last equation by parts, we obtain

$$\frac{d^2I}{db^2} = -\frac{1}{b}\int_0^\infty \frac{\sin u}{u^s}\,\frac{d\cos(b/u)}{du}\,du$$

$$= \frac{1}{b}\int_0^\infty \cos\left(\frac{b}{u}\right)\frac{d}{du}\left(\frac{\sin u}{u^s}\right)du = \frac{J}{b}-\frac{s}{b}\frac{dI}{db}$$

or

$$(3)\qquad \frac{d^2I}{db^2}+\frac{s}{b}\frac{dI}{db}=\frac{J}{b}.$$

Again

$$\frac{dJ}{db} = -\int_0^\infty \cos u\sin\left(\frac{b}{u}\right)\frac{du}{u^{s+1}}.$$

This integral cannot be differentiated again as it stands, the resulting integral not being convergent. But

$$\int_0^\infty (1-\cos u)\sin\left(\frac{b}{u}\right)\frac{du}{u^{s+1}} = \frac{dJ}{db}+\frac{\Gamma(s)}{b^s}\sin\tfrac{1}{2}s\pi$$

may be differentiated with respect to b.

* The truth of this assertion follows easily from the theorems stated by Dr Bromwich, *Infinite Series*, pp. 453 *et seq.*

We thus obtain

$$\frac{d^2J}{db^2} - \frac{\Gamma(s+1)}{b^{s+1}} \sin \tfrac{1}{2}s\pi = \int_0^\infty (1-\cos u) \cos\left(\frac{b}{u}\right) \frac{du}{u^{s+2}}$$

$$= -\frac{1}{b} \int_0^\infty \frac{1-\cos u}{u^s} \frac{d \sin(b/u)}{du} du$$

$$= \frac{1}{b} \int_0^\infty \sin\left(\frac{b}{u}\right) \frac{d}{du}\left(\frac{1-\cos u}{u^s}\right) du$$

$$= \frac{I}{b} - \frac{s}{b} \int_0^\infty \sin\left(\frac{b}{u}\right) \frac{1-\cos u}{u^{s+1}} du$$

$$= \frac{I}{b} - \frac{s}{b} \frac{dJ}{db} - \frac{\Gamma(s+1)}{b^{s+1}} \sin \tfrac{1}{2}s\pi,$$

so that

$$\frac{d^2J}{db^2} + \frac{s}{b} \frac{dJ}{db} = \frac{I}{b}. \tag{3'}$$

The equations (3) and (3)′ may be simplified by putting

$$b = \lambda^{1/(1-s)} = \lambda^{1/\nu}$$

say. We thus obtain

$$\nu\lambda^{1-(1/\nu)} \frac{d}{d\lambda}\left(\nu\lambda^{1-(1/\nu)} \frac{dI}{d\lambda}\right) + \nu(1-\nu)\lambda^{1-(2/\nu)} \frac{dI}{d\lambda} = \lambda^{-(1/\nu)} J$$

or

$$\nu^2\lambda^{2-(1/\nu)} \frac{d^2I}{d\lambda^2} = J, \tag{4}$$

and similarly

$$\nu^2\lambda^{2-(1/\nu)} \frac{d^2J}{d\lambda^2} = I. \tag{4'}$$

From these two equations we can deduce an equation of the fourth order satisfied by both I and J. For

$$\frac{dI}{d\lambda} = \nu(2\nu-1)\lambda^{1-1/\nu} \frac{d^2J}{d\lambda^2} + \nu^2\lambda^{2-1/\nu} \frac{d^3J}{d\lambda^3},$$

$$\frac{d^2I}{d\lambda^2} = (\nu-1)(2\nu-1)\lambda^{-(1/\nu)} \frac{d^2J}{d\lambda^2}$$

$$+ 2\nu(2\nu-1)\lambda^{1-(1/\nu)} \frac{d^3J}{d\lambda^3} + \nu^2\lambda^{2-(1/\nu)} \frac{d^4J}{d\lambda^4},$$

and so

$$(5)\quad \frac{d^4J}{d\lambda^4}+\frac{4\nu-2}{\nu\lambda}\frac{d^3J}{d\lambda^3}+\frac{(\nu-1)(2\nu-1)}{\nu^2\lambda^2}\frac{d^2J}{d\lambda^2}=\frac{\lambda^{(2/\nu)-4}J}{\nu^4}.$$

In the same way we prove that I is also a solution of this equation.

§ 3. The complete integral of this equation* is

$$I=\sqrt{\lambda}\{c_1J^\nu(2\lambda^{1/2\nu})+c_2J^{-\nu}(2\lambda^{1/2\nu})+c_3J^\nu(2i\lambda^{1/2\nu})+c_4J^{-\nu}(2i\lambda^{1/2\nu})\},$$

or

$$I=b^{\frac{1}{2}\nu}\{c_1J^\nu(2\sqrt{b})+c_2J^{-\nu}(2\sqrt{b})+c_3J^\nu(2i\sqrt{b})+c_4J^{-\nu}(2i\sqrt{b});$$

or, if $b=\beta^2$

$$I=\beta^\nu\{c_1J^\nu(2\beta)+c_2J^{-\nu}(2\beta)+c_3J^\nu(2i\beta)+c_4J^{-\nu}(2i\beta)\}.$$

We have now to determine the arbitrary constants. In the first place we observe that

$$I(b)=\int_0^\infty \sin u\sin\left(\frac{\beta^2}{u}\right)\frac{du}{u^{1-\nu}}$$

is small, when β is small, at least of the same order as β^2. For

$$I(b)=\int_0^\pi \sin u\sin\left(\frac{\beta^2}{u}\right)\frac{du}{u^{1-\nu}}+\sin\left(\frac{\beta^2}{\pi}\right)\int_\pi^\xi \frac{\sin u\,du}{u^{1-\nu}},$$

where $\xi>\pi$; or

$$|I|<\beta^2\int_0^\pi \frac{\sin u}{u^{2-\nu}}du+K\sin\left(\frac{\beta^2}{\pi}\right)<K\beta^2.$$

Now, when β is small,

$$\beta^\nu J^\nu(2\beta)\sim\frac{\beta^{2\nu}}{\Gamma(1+\nu)},\quad \beta^\nu J^{-\nu}(2\beta)\sim\frac{1}{\Gamma(1-\nu)},$$

$$\beta^\nu J^\nu(2i\beta)\sim\frac{e^{\frac{1}{2}\nu\pi i}\beta^{2\nu}}{\Gamma(1+\nu)},\quad \beta^\nu J^{-\nu}(2i\beta)\sim\frac{e^{-\frac{1}{2}\nu\pi i}}{\Gamma(1-\nu)};$$

and $0<\nu<1$. Hence we may have

$$c_2+c_4e^{-\frac{1}{2}\nu\pi i}=0,\quad c_1+c_3e^{\frac{1}{2}\nu\pi i}=0.$$

* Nielsen, *Cylinderfunktionen*, p. 138.

Again it is easy to see that, if τ is any positive number, $e^{-\tau\beta} I(b) \to 0$ as $\beta \to \infty$. But, when β is large,

$$J^{\nu}(2i\beta) \sim \frac{e^{2\beta + \frac{1}{2}\nu\pi i}}{2\sqrt{(\pi\beta)}}, \quad J^{-\nu}(2i\beta) \sim \frac{e^{2\beta - \frac{1}{2}\nu\pi i}}{2\sqrt{(\pi\beta)}}.$$

Hence we must have

$$c_3 e^{\frac{1}{2}\nu\pi i} + c_4 e^{-\frac{1}{2}\nu\pi i} = 0. \tag{7}$$

Thus
$$c_1 = -c_2 = -c_3 e^{\frac{1}{2}\nu\pi i} = c_4 e^{-\frac{1}{2}\nu\pi i},$$

and

$$I = C\beta^{\nu}\{J^{\nu}(2\beta) - J^{-\nu}(2\beta) - e^{-\frac{1}{2}\nu\pi i} J^{\nu}(2i\beta) + e^{\frac{1}{2}\nu\pi i} J^{-\nu}(2i\beta)\}. \tag{8}$$

Finally, in order to determine C, we observe that

$$\frac{dI}{d\beta} = 2\beta \int_0^{\infty} \sin u \cos\left(\frac{\beta^2}{u}\right) \frac{du}{u^{2-\nu}},$$

$$\frac{d^2 I}{d\beta^2} = 2\int_0^{\infty} \sin u \cos\left(\frac{\beta^2}{u}\right) \frac{du}{u^{1-\nu}} - 4\beta^2 \int_0^{\infty} \sin u \sin\left(\frac{\beta^2}{u}\right) \frac{du}{u^{3-\nu}}$$

$$= 2\int_0^{\infty} \sin u \cos\left(\frac{\beta^2}{u}\right) \frac{du}{u^{2-\nu}} - 4\int_0^{\infty} \frac{\sin u}{u^{1-\nu}} \frac{d}{du} \cos\left(\frac{\beta^2}{u}\right) du$$

$$= 2\int_0^{\infty} \sin u \cos\left(\frac{\beta^2}{u}\right) \frac{du}{u^{2-\nu}} + 4\int_0^{\infty} \cos\left(\frac{\beta^2}{u}\right) \frac{d}{du}\left(\frac{\sin u}{u^{1-\nu}}\right) du.$$

It is not difficult to prove* that each of these integrals is continuous for $\beta = 0$, so that

$$\lim_{\beta \to 0} \frac{d^2 I}{d\beta^2} = 2\int_0^{\infty} \frac{\sin u}{u^{1-\nu}} du = \frac{\pi}{\Gamma(2-\nu)\sin\frac{1}{2}\nu\pi}.$$

But
$$I = A\beta^2 + B\beta^{2+2\nu} + A'\beta^4 + B'\beta^{4+2\nu} + \ldots,$$

where
$$A = \frac{2C}{\Gamma(2-\nu)}.$$

Hence
$$\frac{4C}{\Gamma(2-\nu)} = \frac{\pi}{\Gamma(2-\nu)\sin\frac{1}{2}\nu\pi}, \quad C = \frac{\pi}{4\sin\frac{1}{2}\nu\pi}.$$

and so finally

$$I(b) = \int_0^{\infty} \sin u \sin\left(\frac{\beta^2}{u}\right) \frac{du}{u^{1-\nu}} \tag{9}$$

$$= \frac{\pi\beta^{\nu}}{4\sin\frac{1}{2}\nu\pi}\{J^{\nu}(2\beta) - J^{-\nu}(2\beta) - e^{-\frac{1}{2}\nu\pi i} J^{\nu}(2i\beta) + e^{\frac{1}{2}\nu\pi i} J^{-\nu}(2i\beta)\},$$

where $\beta = \sqrt{b}$.

* The proof of this, as well as of the legitimacy of the differentiations of § 2, depends on what Dr. Bromwich has called Abel's and Dirichlet's tests for the uniform convergence of infinite integrals: see his *Infinite Series*, pp. 433 *et seq.*

§ 4. The value of $J(b)$ in the general case may be calculated from the equation

$$J(b) = \tfrac{1}{4}\frac{d^2I}{d\beta^2} + \frac{1-2\nu}{4\beta}\frac{dI}{d\beta}.$$

If we observe that $\beta^{\nu}J^{\nu}(2\beta)$ and $\beta^{\nu}J^{-\nu}(2\beta)$ satisfy the equation

$$\frac{d^2u}{d\beta^2} + \frac{1-2\nu}{\beta}\frac{du}{d\beta} = -4u,$$

and $\beta^{\nu}J^{\nu}(2i\beta)$ and $\beta^{\nu}J^{-\nu}(2i\beta)$ satisfy

$$\frac{d^2u}{d\beta^2} + \frac{1-2\nu}{\beta}\frac{du}{d\beta} = 4u,$$

we find without difficulty that

$$(10) \qquad J(b) = \int_0^{\infty} \cos u \cos\left(\frac{\beta^2}{u}\right) \frac{du}{u^{1-\nu}}$$

$$= \frac{\pi\beta^{\nu}}{4\sin\frac{1}{2}\nu\pi}\{-J^{\nu}(2\beta) + J^{-\nu}(2\beta) - e^{-\frac{1}{2}\nu\pi i}J^{\nu}(2i\beta) + e^{\frac{1}{2}\nu\pi i}J^{\nu}(2i\beta)\}.$$

§ 5. The formulæ (9) and (10) have been established on the hypothesis that $0 < \nu < 1$. Changing u into β^2/u, we see that they also hold for $-1 < \nu < 0$. The first integral is convergent if $-2 < \nu < 2$, the second if $-1 < \nu < 1$; and it is not difficult to prove that the formula for the first integral holds throughout its full range of convergence, except for $\nu = -1, 0, 1$.

When $\nu = 0$ the formulæ becomes illusory. The formulæ which replace them may be calculated by a passage to the limit, if we observe that, for small values of ν,

$$J^{\nu}(x) = J^0(x) + \tfrac{1}{2}\pi\nu Y^0(x) + \ldots.^*$$

We thus obtain, for example,

$$(11) \quad \int_0^{\infty} \sin u \sin\left(\frac{\beta^2}{u}\right) \frac{du}{u} = \tfrac{1}{2}\pi\{Y^0(2\beta) + Y^0(2i\beta) + iJ^0(2i\beta)\}.$$

The corresponding formula for the second integral, and the formulæ for the first integral when $\nu = \pm 1$, may also easily be calculated. The formulæ (9), (10), and (11) may be simplified by using a slightly different fundamental system of cylinder-functions; but it is probably preferable to express the integrals in terms of the standard function $J^{\nu}(x)$.

* Nielsen, *Cylinderfunktionen*, pp. 13, 41.

§ 6. The question which is of most interest to us at present is that of the behaviour of the first integral for large values of β. When β is large

$$J^{\nu}(2\beta) = \frac{1+\epsilon_\beta}{\sqrt{(\pi\beta)}} \cos\{2\beta - \tfrac{1}{4}(1+2\nu)\pi\},$$

where $|\epsilon_\beta| < K/\beta$. Hence, when β is large, $J^{\nu}(2\beta) - J^{-\nu}(2\beta)$ behaves like

$$\frac{2\sin\frac{1}{2}\nu\pi}{\sqrt{(\pi\beta)}} \sin(2\beta - \tfrac{1}{4}\pi).$$

On the other hand

$$-e^{-\frac{1}{2}\nu\pi i} J^{\nu}(2i\beta) + e^{\frac{1}{2}\nu\pi i} J^{-\nu}(2i\beta) = -\frac{i}{\sin\nu\pi} e^{\frac{1}{2}\nu\pi i} H_1^{\nu}(2i\beta)^*$$

tends exponentially to zero as $\beta \to \infty$. Hence

$$I(b) \sim \tfrac{1}{2}\sqrt{\pi}\,\beta^{\nu-\frac{1}{2}} \sin(2\beta - \tfrac{1}{4}\pi) = \tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)}\,\beta^{\nu-\frac{1}{2}}(\sin 2\beta - \cos 2\beta).\dagger$$

From this and from § 1 it follows that

$$\int_0^{\xi} \sin\left(\frac{m}{x}\right) \sin\lambda x \frac{dx}{x^{1+t}} \sim \tfrac{1}{2}\sqrt{\pi}\, m^{-\frac{1}{2}t-\frac{1}{4}} \lambda^{\frac{1}{2}t-\frac{1}{4}} \sin\{2\sqrt{(\lambda m)} - \tfrac{1}{4}\pi\}.$$

Thus, *if* $f(x) = x^{-t}\sin(m/x)$, *the Dirichlet's Integral*

$$J(\lambda) = \int_0^{\xi} f(x) \frac{\sin\lambda x}{x}\, dx$$

tends to zero as $\lambda \to \infty$ *if* $t < \frac{1}{2}$, *oscillates finitely if* $t = \frac{1}{2}$, *oscillates infinitely if* $t > \frac{1}{2}$.

§ 7. It will be observed that there is an essential difference between this particular result and the results proved in Notes XXIII. and XXIV. for the case in which $\psi(1/x)$ is not higher in the logarithmic scale than $\log(1/x)$. Then, to put the matter roughly, the boundary of oscillation for $J(\lambda)$ was the same as that for $f(x)$. If $f(x)$ oscillated as $x \to \infty$, so did $J(\lambda)$ as $\lambda \to \infty$, and not otherwise. But in this instance all the cases, for which $0 \leq t < \frac{1}{2}$, are cases in which $f(x)$ oscillates and $J(\lambda) \to 0$.

§ 8. This result is, of course, in agreement with Du Bois Reymond's general conclusions.‡ Du Bois Reymond enunci-

* Nielsen, *Cylinderfunktionen*, p. 17.

† Here, as in Note XXIV., the symbol ~ is used in a sense rather wider than usual: we mean,

$$I(b) = \tfrac{1}{2}\sqrt{\pi}\,\beta^{\nu-\frac{1}{2}}\{\sin(2\beta - \tfrac{1}{4}\pi) + \epsilon_\beta\},$$

and not

$$I(b) = \tfrac{1}{2}\sqrt{\pi}\,\beta^{\nu-\frac{1}{2}}(1+\epsilon_\beta)\sin(2\beta - \tfrac{1}{4}\pi).$$

‡ Du Bois Reymond, *Abhandlungen d. Bay. Ak.*, Bd. XII. (2), p. 37. See also Reimann, *Gesammelte Werke*, pp. 246–8.

ated the following general result: the boundary of oscillation of $J(\lambda)$, when

$$f(x) = \rho(x) \sin \sigma(x),$$

and $\sigma(x)$ is higher in the logarithmic scale than $\log(1/x)$, is given by

$$\rho(x) \sim x \sqrt{|\sigma''(x)|}.$$

If $\sigma(x) = m/x$, this gives $\rho(x) \sim 1/\sqrt{x}$, in agreement with our previous conclusions.

I hope before long to publish elsewhere a rigorous and reasonably simple proof of results comprising and making more precise all, or nearly all, that Du Bois Reymond proved or stated in his memoir referred to above. For this Notes XXIII., XXIV., and XXVII. may be regarded as preliminary studies. The whole investigation, I find, will run to too great a length to be included in this series.

Correction to Note XXIV.

I TAKE this opportunity of correcting two inaccuracies in the argument of Note XXIV.

The equation (3) on p. 177 is not quite accurate: the factors

$$(1 + \epsilon) \sin \psi(u)$$

should be replaced by the single factor $\sin \psi(u) + \epsilon$.

We have, in fact, as appears at once upon differentiation,

$$\theta'(u) = (\alpha + 1) u^{\alpha} L(u) (1 + \epsilon) \sin \psi(u) + u^{\alpha} L(u) \epsilon \cos \psi(u)$$
$$= (\alpha + 1) u^{\alpha} L(u) \{\sin \psi(u) + \epsilon\},$$

where $L(u) = (\log u)^{\alpha_1} (\log \log u)^{\alpha_2}$, and ϵ, in each place where it occurs, denotes a function whose limit, as $u \to \infty$, is zero.

We divide up the integral into the three parts $I_1(\lambda)$, $I_2(\lambda)$, $I_3(\lambda)$, as in Note XXIV. We have

$$|\theta'(u)| < K u^{\alpha} L(u),$$

and we have first to show that:

(i) if $\lambda\delta \to 0$, then

$$\int_0^{\delta} x^{-\alpha} L\left(\frac{1}{x}\right) \left(\frac{\sin \frac{1}{2}\lambda x}{\frac{1}{2}\lambda x}\right)^3 \tfrac{1}{2}\lambda dx \Big/ \lambda^{\alpha} L(\lambda) \to 0,$$

(ii) if $\lambda\delta \to \infty$, then

$$\int_{\delta}^{\xi} x^{-\alpha} L\left(\frac{1}{x}\right) \left(\frac{\sin \frac{1}{2}\lambda x}{\frac{1}{2}\lambda x}\right)^3 \tfrac{1}{2}\lambda dx \Big/ \lambda^{\alpha} L(\lambda) \to 0,$$

The proof (pp. 178–9) of the first of these relations is again not quite accurate. We have, indeed,

$$\left(\log\frac{\lambda}{2t}\right)^{\alpha_1} < K(\log\lambda)^{\alpha_1}\left(\log\frac{1}{t}\right)^{\alpha_1},$$

$$\left(\log\log\frac{\lambda}{2t}\right)^{\alpha_2} < K(\log\log\lambda)^{\alpha_2}\left(\log\log\frac{1}{t}\right)^{\alpha_2},$$

if both α_1 and α_2 are positive; but if either is negative the corresponding inequality must be replaced by one of

$$\left(\log\frac{\lambda}{2t}\right)^{\alpha_1} < (\log\lambda)^{\alpha_1}, \quad \left(\log\log\frac{\lambda}{2t}\right)^{\alpha_2} < (\log\log\lambda)^{\alpha_2}.$$

We have in fact

$$L\left(\frac{\lambda}{2t}\right) < L(\lambda) \text{ or } < KL(\lambda)\,L\left(\frac{1}{t}\right),$$

according as $L(\lambda) \to 0$ or ∞. The proof of (i) may then be completed as in Note XXIV. Similarly, if $\frac{1}{2}\lambda\delta < t < \frac{1}{2}\lambda\xi$, we have

$$L\left(\frac{\lambda}{2t}\right) < L(\lambda) \text{ or } < KL(\lambda)/L(t),$$

according as $L(\lambda) \to \infty$ or 0; and then the proof of (ii) may be completed similarly. It follows that

$$I(\lambda) = I_2(\lambda) + J(\lambda),$$

where $J(\lambda)/\lambda^a L(\lambda) \to 0$, provided only $\mu \to \infty$, no matter how slowly. Again

$$I_2(\lambda) = I_2' + I_2'',$$

where $$I_2' = (a+1)\int_{\delta_1}^{\delta_2} u^a L(u)\sin\psi(u)\left(\frac{\sin\frac{1}{2}\lambda x}{\frac{1}{2}\lambda x}\right)^2 \tfrac{1}{2}\lambda dx,$$

$$|I_2''| < K\int_{\delta_1}^{\delta_2} u^a L(u)\,|\epsilon_u|\left(\frac{\sin\frac{1}{2}\lambda x}{\frac{1}{2}\lambda x}\right)^2 \tfrac{1}{2}\lambda dx.$$

If now we apply to these integrals the argument of pp. 177-8, we find that

$$I_2' = (a+1)\lambda^a L(\lambda)\{\sin\psi(\lambda) + \epsilon_\lambda\}\int_0^\infty t^{-a}\left(\frac{\sin t}{t}\right)^2 dt,$$

and that $$I_2''/\lambda^a L(\lambda) \to 0.$$

Thus, finally, we find

$$I(\lambda) = -\Gamma(-a)\sin\tfrac{1}{2}a\pi\,\lambda^a L(\lambda)\{\sin\psi(\lambda) + \epsilon_\lambda\}.$$

We have supposed $0<a<1$. The argument remains valid when $a=0$, if we replace $-\Gamma(-a)\sin\frac{1}{2}a\pi$ by its limit $\frac{1}{2}\pi$. When $a=1$ the argument fails and the formula is plainly invalid: in this case we have to suppose certain conditions satisfied by α_1 and α_2 to ensure the convergence of the integral.

The fact that we cannot deal in this way with the case in which $a=1$ has led me, in the more extended investigation to which I referred at the end of Note XXVII., to adopt a slightly different method.

A CLASS OF DEFINITE INTEGRALS.

By *G. H. Hardy*, Trinity College, Cambridge.

THERE is one case in which the values of the integrals considered in the preceding note may be expressed in terms of elementary functions. If we put

$$s=\nu=\tfrac{1}{2},$$

and observe that

$$J^{\frac{1}{2}}(2\beta)=\frac{\sin 2\beta}{\sqrt(\pi\beta)}, \quad J^{-\frac{1}{2}}(2\beta)=\frac{\cos 2\beta}{\sqrt(\pi\beta)},$$

$$J^{\frac{1}{2}}(2i\beta)=e^{\frac{1}{4}\pi i}\frac{\sinh 2\beta}{\sqrt(\pi\beta)}, \quad J^{-\frac{1}{2}}(2i\beta)=e^{-\frac{1}{4}\pi i}\frac{\cosh 2\beta}{\sqrt(\pi\beta)},$$

we obtain

$$I(h)=\tfrac{1}{2}\sqrt(\tfrac{1}{2}\pi)\,(e^{-2\beta}-\cos 2\beta+\sin 2\beta).$$

In this case also

$$J=\frac{1}{4}\frac{d^2I}{d\beta^2},$$

so that we obtain the formulæ

$$(1)\quad \int_0^\infty \sin x\sin\left(\frac{\beta^2}{x}\right)\frac{dx}{\sqrt{x}}=\tfrac{1}{2}\sqrt(\tfrac{1}{2}\pi)\,(e^{-2\beta}-\cos 2\beta+\sin 2\beta),$$

$$(2)\quad \int_0^\infty \cos x\cos\left(\frac{\beta^2}{x}\right)\frac{dx}{\sqrt{x}}=\tfrac{1}{2}\sqrt(\tfrac{1}{2}\pi)\,(e^{-2\beta}+\cos 2\beta-\sin 2\beta).$$

By changing x into β^2/x, we deduce

$$(3)\quad \int_0^\infty \sin x\sin\left(\frac{\beta^2}{x}\right)\frac{dx}{x\sqrt{x}}=\frac{1}{2\beta}\sqrt(\tfrac{1}{2}\pi)\,(e^{-2\beta}-\cos 2\beta+\sin 2\beta),$$

$$(4)\quad \int_0^\infty \cos x\cos\left(\frac{\beta^2}{x}\right)\frac{dx}{x\sqrt{x}}=\frac{1}{2\beta}\sqrt(\tfrac{1}{2}\pi)\,(e^{-2\beta}+\cos 2\beta-\sin 2\beta).$$

Differentiating the two former equations with respect to β, we obtain

$$(5)\quad \int_0^\infty \sin x \cos\left(\frac{\beta^2}{x}\right)\frac{dx}{x\sqrt{x}} = -\frac{1}{2\beta}\sqrt{(\tfrac{1}{2}\pi)}(e^{-2\beta}-\sin 2\beta-\cos 2\beta),$$

$$(6)\quad \int_0^\infty \cos x \sin\left(\frac{\beta^2}{x}\right)\frac{dx}{x\sqrt{x}} = \frac{1}{2\beta}\sqrt{(\tfrac{1}{2}\pi)}(e^{-2\beta}+\sin 2\beta+\cos 2\beta),$$

Changing x into β^2/x, we obtain

$$(7)\quad \int_0^\infty \cos x \sin\left(\frac{\beta^2}{x}\right)\frac{dx}{\sqrt{x}} = -\tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)}(e^{-2\beta}-\sin 2\beta-\cos 2\beta),$$

$$(8)\quad \int_0^\infty \sin x \cos\left(\frac{\beta^2}{x}\right)\frac{dx}{\sqrt{x}} = \tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)}(e^{-2\beta}+\sin 2\beta+\cos 2\beta),$$

Combining these results by addition and subtraction, we obtain the following eight additional formulæ:

$$(9)\quad \int_0^\infty \cos\left(x+\frac{\beta^2}{x}\right)\frac{dx}{\sqrt{x}} = \sqrt{(\tfrac{1}{2}\pi)}(\cos 2\beta-\sin 2\beta),$$

$$(10)\quad \int_0^\infty \cos\left(x-\frac{\beta^2}{x}\right)\frac{dx}{\sqrt{x}} = \sqrt{(\tfrac{1}{2}\pi)}\,e^{-2\beta},$$

$$(11)\quad \int_0^\infty \sin\left(x+\frac{\beta^2}{x}\right)\frac{dx}{\sqrt{x}} = \sqrt{(\tfrac{1}{2}\pi)}(\cos 2\beta+\sin 2\beta),$$

$$(12)\quad \int_0^\infty \sin\left(x-\frac{\beta^2}{x}\right)\frac{dx}{\sqrt{x}} = \sqrt{(\tfrac{1}{2}\pi)}\,e^{-2\beta},$$

$$(13)\quad \int_0^\infty \cos\left(x+\frac{\beta^2}{x}\right)\frac{dx}{x\sqrt{x}} = \frac{1}{\beta}\sqrt{(\tfrac{1}{2}\pi)}(\cos 2\beta-\sin 2\beta),$$

$$(14)\quad \int_0^\infty \cos\left(x-\frac{\beta^2}{x}\right)\frac{dx}{x\sqrt{x}} = \frac{1}{\beta}\sqrt{(\tfrac{1}{2}\pi)}\,e^{-2\beta},$$

$$(15)\quad \int_0^\infty \sin\left(x+\frac{\beta^2}{x}\right)\frac{dx}{x\sqrt{x}} = \frac{1}{\beta}\sqrt{(\tfrac{1}{2}\pi)}(\cos 2\beta+\sin 2\beta),$$

$$(16)\quad \int_0^\infty \sin\left(x-\frac{\beta^2}{x}\right)\frac{dx}{x\sqrt{x}} = -\frac{1}{\beta}\sqrt{(\tfrac{1}{2}\pi)}\,e^{-2\beta}.$$

Some, at any rate, of these results are originally due to Cauchy, and some are to be found in Bierens de Haan's *Tables*. I have set them out at length here, because I do not know where else they are systematically stated.

ON A THEOREM OF TANNERY'S AND ITS ANALOGUE IN THE THEORY OF DEFINITE INTEGRALS.

By *W. H. Young, Sc.D., F.R.S.*

§ 1. As I have had occasion more than once to point out, no necessary and sufficient conditions, in the proper sense of the words, for the existence of an unique double limit are known. It is therefore desirable to lay stress on each special case in which the existence can be asserted. In the present Note I begin accordingly by showing what I have already virtually proved elsewhere* that when the function is monotone with respect to one of the variables, the existence of an unique double limit for modes of approach to the origin in, say, one of the four quadrants, is an immediate consequence of the equality of the two repeated limits, supposing these to be equal. I then proceed to apply this fact to prove the following theorem:—

If $f(x, y)$ is a bounded function of (x, y), then $\int_a^X f(x, y)\,dx$ *has, as X and y each approach the value ∞, an unique double limit, provided only two other bounded functions $g(x, y)$, $h(x, y)$ can be found, having this property, and such that*

$$g(x, y) \leq f(x, y) \leq h(x, y),$$

all three functions being supposed to have USUALLY† *unique limits as y becomes infinite.*

This theorem, of which a still more general form is also proved, leads at once to a generalisation of a theorem called by Bromwich‡ an analogue of Tannery's theorem, or, to speak more correctly, removes from the statement of that theorem an unnecessary restriction, namely, the restriction that $f(x, y)$ should, as y approaches the value ∞, approach its limit uniformly.

In the proof I make use of a result obtained recently by myself in a paper entitled "Semi-integrals and Oscillating Successions of Functions," presented to the London Mathe-

* "On Uniform Oscillation" (1909), *Camb. Phil. Trans.*, vol. XXI, p. 255.

† USUALLY means here "except possibly for a set of values of x of content zero."

‡ T. J. I'a Bromwich, *An Introduction to the Theory of Infinite Series* (1908), Macmillan & Co., p. 438.

matical Society, namely, that integration over a finite or infinite interval is allowable for a sequence of functions $f_1, f_2, \ldots$ if two other sequences $g_1, g_2, \ldots$ and $h_1, h_2, \ldots$ can be found, for which this is the case, and which are such that

$$g_n < f_n < h_n.$$

I terminate the note by giving an extension of Tannery's original theorem. The proof employed is of an intuitive character, and is, like the other proofs of this paper, based on the method of comparison.

§ 2. *Lemma* 1. *If $f(x, y)$ is monotone with respect to x, and if at the origin the repeated limits are unique and equal, viz.,*

$$\operatorname*{Lt}_{x=0}\ \operatorname*{Lt}_{y=0} f(x, y) = \operatorname*{Lt}_{y=0}\ \operatorname*{Lt}_{x=0} f(x, y),$$

then $f(x, y)$ has at the origin an unique double limit

$$\operatorname*{Lt}_{\substack{x=0\\y=0}} f(x, y).$$

For, if $f(x, y)$ is monotone increasing with respect to x,

$$\operatorname*{Lt}_{x=0} f(x, y) \leq f(x, y),$$

supposing x and y both to be positive. Hence, letting (x, y) approach $(0, 0)$ in any manner so as to give an unique limit U,

$$\operatorname*{Lt}_{y=0}\ \operatorname*{Lt}_{x=0} f(x, y) \leq U \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Moreover, fixing any value of x, say x_0, we have, for all smaller values of x,

$$f(x, y) \leq f(x_0, y),$$

and therefore, letting (x, y) move along our chosen sequence*,

$$U \leq \operatorname*{Lt}_{y=0} f(x_0, y),$$

whence, letting x_0 approach zero as limit,

$$U \leq \operatorname*{Lt}_{x=0}\ \operatorname*{Lt}_{y=0} f(x, y) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

* It is here apparently assumed that the single limit is unique when y approaches zero; this is usually understood to be the case when the repeated limits exist. If, however, the assumption is interpreted in the more general sense in which the single limit is not necessarily unique, the above reasoning still holds, using the upper limit of $f(x, y)$ when y approaches zero, instead of the unique limit.

From (1) and (2) we see that U is equal to the two repeated limits, which, by hypothesis, are equal. Since U is any one of the double limits, this proves the theorem when $f(x, y)$ is monotone increasing, and similarly if it is monotone decreasing, with respect to x.

Similarly, if the point (x, y) lies in one of the other quadrants, the result follows as here for the $(+, +)$-quadrant.

§ 3. *Lemma 2. If $f(x, y)$ is a positive* bounded function of (x, y) for which the integral $\int_a^X f(x, y)\,dx$ has an unique double limit U when X and y each approach the value ∞ in any manner whatever, then the same is true if we change f into any smaller positive function $g(x, y)$; f and g having* USUALLY *unique limits as y becomes infinite.*

For, since the double limit is unique, the repeated limits exist and are equal. Therefore

$$\operatorname*{Lt}_{y=\infty}\operatorname*{Lt}_{X=\infty}\int_a^X f(x, y)\,dx = \operatorname*{Lt}_{X=\infty}\operatorname*{Lt}_{y=\infty}\int_a^X f(x, y)\,dx.$$

But, since $f(x, y)$ is bounded both above and below, we may† replace $\operatorname*{Lt}_{y=\infty}\int_a^X f(x, y)\,dx$ by $\int_a^X \operatorname*{Lt}_{y=\infty} f(x, y)\,dx$. The preceding equation then becomes

$$\operatorname*{Lt}_{y=\infty}\int_a^\infty f(x, y)\,dx = \int_a^\infty \operatorname*{Lt}_{y=\infty} f(x, y)\,dx.$$

Hence,‡ if $g(x, y)$ is any positive function $\leq f(x, y)$, the same is true if we replace f by g.

Now, since $g(x, y)$ is also bounded, we may reverse our steps, and we get

$$\operatorname*{Lt}_{y=\infty}\operatorname*{Lt}_{X=\infty}\int_a^X g(x, y)\,dx = \operatorname*{Lt}_{X=\infty}\operatorname*{Lt}_{y=\infty}\int_a^X g(x, y)\,dx.$$

But, since $g(x, y)$ is a positive function, $\int_a^X g(x, y)\,dx$ is a monotone function of X, so that it satisfies the conditions of the preceding Lemma, whence the result stated at once follows.

* That is ≥ 0.

† See "Term by Term Integration of Oscillating Series," *Proc. L.M.S.* (1909), series 2, vol. 8, p. 111.

‡ By the corollary to the theorem quoted in § 1.

Corollary. The same is true if, without being bounded, $f(x, y)$ is positive, and such that

$$\operatorname*{Lt}_{y=\infty} \int_a^X f(x, y)\, dx = \int_a^X \operatorname*{Lt}_{y=\infty} f(x, y)\, dx,$$

e.g., if $f(x, y)$ is a monotone function of y.

§4. We can now prove the theorem stated in the introduction.

THEOREM. *If $f(x, y)$ is a bounded function of (x, y), then $\int_a^X f(x, y)\, dx$ has, as X and y each approach the value ∞ in any manner, an unique double limit, provided two other bounded functions of (x, y) can be found, say g and h, having this property and such that*

$$g(x, y) \leq f(x, y) \leq h(x, y),$$

all three functions being supposed to have USUALLY *unique limits when y becomes infinite.*

For $h(x, y) - g(x, y)$ is then a function satisfying the conditions of Lemma 2, and $f(x, y) - g(x, y)$ is a smaller positive function. Hence, by Lemma 2,

$$\int_a^X (f - g)\, dx$$

has an unique double limit when X and y approach infinity in any manner whatever. Since $\int_a^X g(x, y)\, dx$ has also an unique double limit, the same is true of the sum of the last two integrals, that is of $\int_a^X f(x, y)\, dx$, which proves the theorem.

Corollary 1. *The same is true if, without f, g, and h being bounded, g and h are such as to satisfy the condition of the corollary in the preceding article, e.g., if g and h are monotone functions of y.*

We have in this case to use the theorem quoted from "Semi-Integrals," not only for an infinite but also for a finite interval.

Corollary 2. *The double limit in question is unique if we can determine a positive function $M(x)$ such that $|f(x, y)| < M(x)$, for all values of y, while $\int_a^\infty M(x)\, dx$ converges.*

This is precisely equivalent to Bromwich's statement on p. 438, *loc. cit.*, except that the requirement that $f(x, y)$ should converge uniformly to its limit when y is infinite is here omitted.

Corollary 3. *If $f(x, y)$ is a positive and monotone decreasing function of x, then the double limit exists for the integral, when the integrand is $f(x, y)\, g(x)$, provided that $g(x)$ has an absolutely convergent improper integral in the interval (a, ∞).*

In fact, it will clearly be sufficient to prove the theorem under the hypothesis that $g(x)$ is positive. But in this case the integrand satisfies the conditions of the preceding corollary.*

§ 5. The theorem known as Tannery's theorem, mentioned in the introduction, is stated as follows by Bromwich :†

Suppose we are given a sum

$$F(n) = v_0(n) + v_1(n) + \ldots + v_p(n),$$

and that we want to find the limit $\underset{n=\infty}{\mathrm{Lt}}\ F(n)$, *it being understood that p tends steadily to infinity with n. Then, if we have*

$$\underset{n=\infty}{\mathrm{Lt}}\ v_r(n) = w_r \quad (r \text{ being fixed}),$$

the limit is given by

$$\underset{n=\infty}{\mathrm{Lt}}\ F(n) = w_0 + w_1 + w_2 + \ldots \text{to } \infty = W,$$

provided that $|v_r(n)| < M_r$, where M_r is independent of n, and the series ΣM_r is convergent.

It will be noticed that this virtually asserts the uniqueness of a certain double limit. In fact, if, for convenience, we write $F(n, p)$ for $F(n)$, it asserts that, under the conditions stated, however n and p tend simultaneously to infinity, we

* It should be noted that, in the very important corresponding theorem proved for functions $g(x)$ which have non-absolutely convergent integrals on p. 113 of Bromwich's book, the tacit assumption that $g(x)$ is bounded, is made, while the explicit assumption there made that $f(x, y)$ should approach its limit uniformly is unnecessary, in view of the fact that a sequence which is bounded can always be integrated.

† *loc. cit.*, p. 123.

get the same result as if we first make n tend to infinity, and then p.

I propose now to prove the following theorem, which evidently includes the above as a particular case :—

THEOREM. *If*

$$F(n, p) = f_0(n) + f_1(n) + \ldots + f_p(n),$$

then $F(n, p)$ has an unique double limit, when n and p simultaneously approach infinity in any manner, provided only we can find two other sets of functions g_r and h_r, such that, for all values of r,

$$g_r(n) \leq f_r(n) \leq h_r(n),$$

while the corresponding summations $G(n, p)$ and $H(n, p)$ have unique double limits.

Here, we suppose $f_r(n)$, $g_r(n)$, and $h_r(n)$ to have unique limits when n approaches infinity, r being fixed.

For, if a function of (n, p) is monotone increasing with respect to p, all its double limits are greater than, or equal to, its repeated limit, when first n and afterwards p becomes infinite. Hence, bearing in mind that a repeated limit is a double limit,

$$\operatorname*{Lt}_{\substack{n=\infty \\ p=\infty}} (H - F) > \operatorname*{Lt}_{p=\infty} \operatorname*{Lt}_{n=\infty} (H - F)$$

and

$$\operatorname*{Lt}_{\substack{n=\infty \\ p=\infty}} (F - G) \geq \operatorname*{Lt}_{p=\infty} \operatorname*{Lt}_{n=\infty} (F - G).$$

But, since H and G have unique double limits, so has the sum $H - G$ of the two functions which occur in these relations, and this double limit is the sum of the two unique repeated limits on the right in them. Hence the relations must both be equalities, not inequalities. For, if there was any sequence of ensembles (n, p) yielding a double limit for $H - F$ which was greater than $\operatorname*{Lt}_{p=\infty} \operatorname*{Lt}_{n=\infty} (H - F)$, and we pick out a subsequence giving a double limit for $F - G$ which satisfies, of course, the second of the above relations, we should get passing along this subsequence the sum of these two double limits as a double limit of $(H - G)$ greater than its unique double limit, which is impossible. Therefore $F - G$ has an unique double limit, whence, since G has an unique double

limit, the same is true of F. Moreover the unique double limit of F is plainly finite, since it is not greater than that of $H(n, p)$, which is assumed to be finite.

Corollary. If

$$F(n, p) = \{1 + f_0(n)\}\{1 + f_1(n)\}\ldots\{1 + f_p(n)\},$$

and if $G(n, p)$ and $H(n, p)$ denote the products obtained by changing f into g and h respectively, where, for all values of r,

$$g_r(n) \leq f_r(n) \leq h_r(n),$$

all three functions having unique limits as n becomes infinite, then $F(n, p)$ has an unique double limit, when n and p simultaneously approach infinite in any manner, provided this is true for both $G(n, p)$ and $H(n, p)$. We prove this by taking logarithms. A special case of this corollary is the theorem stated at the bottom of p. 124 of Bromwich's book.

§ 6. It is remarkable that, while in the theory of convergence pure and simple, the method of comparison has been the one usually employed, there is no trace elsewhere of the systematic employment of such a method, even in the treatment of the theory of uniform convergence, though this theory is almost as simple, and though the theorems required are then almost self-evident.

The reader who is interested in seeing the possibilities of this method of comparison in the more difficult theories of term-by-term differentiation and integration of series and of definite integrals may consult besides the paper on semi-integrals above cited another forthcoming paper by the author on such applications.*

* Weierstrass's test would seem to be the only exception to this rule; the series moreover with which, in the application of that test, the series under discussion is compared is one with constant terms only.

Geneva,
July 15th, 1910.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XXVIII.

A conditionally convergent double integral.

§ 1. The value of the integral

$$\int_{-\infty}^{\infty} \frac{e^{lxi}}{k+axi}\,dx, \tag{1}$$

where l is real and positive, and

$$k=\kappa+\kappa' i, \quad a=\alpha+\alpha' i$$

was proved by Cauchy and Dirichlet to be

$$\frac{2\pi}{a}\,e^{-kl/a},$$

or zero, according as the imaginary part of ik/a is positive or negative, that is to say according as $\kappa\alpha+\kappa'\alpha' \gtrless 0$. If $\kappa\alpha+\kappa'\alpha'=0$ the integral is not convergent, but its principal value is determinate and equal to

$$\frac{\pi}{a}\,e^{-kl/a}.$$

This result naturally suggests the consideration of the double integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{(lx+my)i}}{k+(ax+by)i}\,dx\,dy, \tag{2}$$

where l and m are real, and

$$k=\kappa+\kappa' i, \quad a=\alpha+\alpha' i, \quad b=\beta+\beta' i.$$

This integral is, in fact, another interesting member of the class of conditionally convergent double integrals that I have considered in several of this series of notes.

§ 2. We must first consider the question of the convergence of the integral (2).

The denominator of the subject of integration will vanish if

$$\kappa - \alpha' x - \beta' y = 0, \quad \kappa' + \alpha x + \beta y = 0,$$

or if $$x = -\frac{\kappa\beta + \kappa'\beta'}{\alpha\beta' - \alpha'\beta}, \quad y = \frac{\kappa\alpha + \kappa'\alpha'}{\alpha\beta' - \alpha'\beta},$$

or if $$x = -(\kappa\beta + \kappa'\beta')/\Delta = x_0, \quad y = (\kappa\alpha + \kappa'\alpha')/\Delta = y_0,$$

say. We shall suppose for the present that $\Delta = \alpha\beta' - \alpha'\beta \neq 0$; the case in which $\Delta = 0$ is clearly exceptional. Moreover

$$k + (ax + by)\, i = ai\,(x - x_0) + bi\,(y - y_0)$$

or, if $$x - x_0 = \rho\cos\phi, \quad y - y_0 = \rho\sin\phi,$$

$$k + (ax + by)\, i = (i\alpha - \alpha')\,\rho\cos\phi + (i\beta - \beta')\,\rho\sin\phi,$$

an expression whose modulus is equal to the product of ρ by an essentially positive function of $\cos\phi$ and $\sin\phi$. We conclude, by a transformation to polar coordinates (ρ, ϕ), whose origin is the point (x_0, y_0), that the integral converges absolutely over that point.

The only doubt as to the convergence of the integral that can arise is therefore that suggested by the infinite limits. In the first place the integral is not *absolutely* convergent, since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{|k + (ax + by)\, i|}$$

is clearly divergent. But, if $\Delta \neq 0$, the integral is conditionally convergent in the sense that

$$\int_{-H'}^{H}\int_{-K'}^{K}$$

tends to a definite limit as H, H', K, and K' tends simultaneously and independently to ∞. I shall prove this by a method which I have used in connection with some other classes of double integrals,* and which at the same time gives the value of the integral.

* See, *e.g.*, *Messenger*, vol. XXXII., p. 159, and *Quarterly Journal*, vol. XXXVII., p. 360.

§ 3. I define the region D' as consisting of the interior of the rectangle

$$-H' < x < H, \quad -K' < y < K,$$

except the interior of a small circle whose centre is (x_0, y_0) and whose radius is ρ. To this region I apply the formula

$$(3) \qquad \int (\lambda F + \mu G)\, ds = \iint \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) dx\, dy,$$

where λ, μ are the direction cosines of the outward normal to the boundary of D', and

$$F = \frac{bi}{am - bl} \frac{e^{(lx+my)i}}{k + (ax+by)\, i},$$

$$G = -\frac{ai}{am - bl} \frac{e^{(lx+my)i}}{k + (ax+by)\, i},$$

so that

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = \frac{e^{(lx+my)i}}{k + (ax+by)\, i}.$$

It should be observed that $am - bl$ cannot be zero, since this would involve

$$\alpha m - \beta l = 0, \quad \alpha' m - \beta' l = 0,$$

and so $\Delta = 0$.

We have therefore

$$(4) \qquad \int_{(D')} \frac{e^{(lx+my)i}}{k + (ax+by)\, i} dx\, dy = \int (\lambda F + \mu G)\, ds,$$

the line integral being taken round the boundary of D'.

§ 4. It is easy to prove that the integral

$$\int (\lambda F + \mu G)\, ds,$$

taken along any one of the sides of the rectangle, ultimately vanishes. Consider, for example, the side for which

$$x = H, \quad -K' \leq y \leq K, \quad \lambda = 1, \quad \mu = 0, \quad ds = dy.$$

We have to prove that

$$\int_{-K'}^{K} \frac{e^{(lH+my)i}\, dy}{k + (aH+by)\, i} \to 0$$

when H, K, and K' tend in any way to ∞. It is evident that we may neglect the factor $\exp(lHi)$ and replace K and K' by

$$2n\pi/m, \quad 2n'\pi/m,$$

the largest integral multiples of $2\pi/m$ contained in them.

Then

$$\int_0^{2n\pi/m} \frac{e^{myi}\,dy}{k+(aH+by)\,i} = \int_0^{2n\pi} \frac{e^{ui}\,du}{K+bui} = \int_0^{\pi} \Theta e^{ui}\,du,$$

where $K = m(k+aHi)$, and

$$\Theta = \frac{1}{K+bui} - \frac{1}{K+b(u+\pi)\,i} + \frac{1}{K+b(u+2\pi)\,i} - \cdots$$

$$+ \frac{1}{K+b\{u+(2n-2)\,\pi\}\,i} - \frac{1}{K+b\{u+(2n-1)\,\pi\}\,i}$$

$$= \sum_{\nu=0}^{n-1} \left[\frac{1}{K+b(u+2\nu\pi)\,i} - \frac{1}{K+b\{u+(2\nu+1)\,\pi\}\,i} \right]$$

$$= b\pi i \sum_{\nu=0}^{n-1} \frac{1}{[K+b(u+2\nu\pi)\,i]\,[K+b\{u+(2\nu+1)\,\pi\}\,i]},$$

the modulus of which is less than a constant multiple of

$$\sum_{\nu=0}^{\infty} \frac{1}{(|K|+2\nu)(|K|+2\nu+1)} = \frac{1}{|K|}.$$

It follows at once that

$$\int_0^{2n\pi/m} \frac{e^{myi}\,dy}{k+(aH+by)\,i} \to 0.$$

Similarly the integral from $-2n'\pi/m$ to 0 tends to zero; and so the line integral along the side under consideration vanishes in the limit. A similar proof of course applies to the other sides of the rectangle.

§ 5. It remains to consider the contribution to the line integral of the small circle. Along the circle we have

$$x - x_0 = \rho\cos\phi, \quad y - y_0 = \rho\sin\phi,$$

$$\lambda = -\cos\phi, \quad \mu = -\sin\phi, \quad ds = \rho d\phi,$$

and the value of the integral is in the limit*

$$\frac{e^{(lx_0+my_0)i}}{am-bl}\int_0^{2\pi}\frac{a\sin\phi-b\cos\phi}{a\cos\phi+b\sin\phi}\,d\phi$$

$$=-\frac{e^{(lx_0+my_0)i}}{am-bl}\Big[\log(a\cos\phi+b\sin\phi)\Big]_0^{2\pi}.$$

The value of the last factor is some multiple of $2\pi i$. Now the imaginary part of the logarithm is

$$\arctan\left(\frac{\alpha'\cos\phi+\beta'\sin\phi}{\alpha\cos\phi+\beta\sin\phi}\right),$$

and the differential coefficient of this expression is

$$\frac{\alpha\beta'-\alpha'\beta}{(\alpha\cos\phi+\beta\sin\phi)^2+(\alpha'\cos\phi+\beta'\sin\phi)^2};$$

which has the sign of Δ. Also, as ϕ increases from 0 to 2π, $\alpha\cos\phi+\beta\sin\phi$ vanishes twice. Hence the increment of the logarithm is

$$2\pi i\,\mathrm{sgn}\,\Delta.$$

Hence, finally,

$$(5)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{(lx+my)i}\,dx\,dy}{k+(ax+by)\,i}=\frac{2\pi\,\mathrm{sgn}\,\Delta}{(am-bl)\,i}e^{(lx_0+my_0)i},$$

where $\qquad k=\kappa+\kappa' i,\quad a=\alpha+\alpha' i,\quad b=\beta+\beta' i,$

$$\Delta=\alpha\beta'-\alpha'\beta,\quad x_0=-(\kappa\beta+\kappa'\beta')/\Delta,\quad y_0=(\kappa\alpha+\kappa'\alpha')/\Delta.$$

§ 6. It is clear that the preceding analysis may be applied to the more general integral

$$(6)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\phi(lx+my)\,dx\,dy}{k+(ax+by)\,i}$$

in a large variety of cases. We suppose that $\Phi(u)$ is an integral function of $\phi(u)$, the constant of integration being

* It is hardly worth while to set out the details of the proof.

fixed by a suitable specification as to the behaviour of $\Phi(u)$ for large values of u, and we take

$$F = -\frac{b}{am-bl}\frac{\Phi(lx+my)}{k+(ax+by)i},$$

$$G = \frac{a}{am-bl}\frac{\Phi(lx+my)}{k+(ax+by)i},$$

so that

$$\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=\frac{\Phi(lx+my)}{k+(ax+by)i};$$

and, provided the line integrals along the sides of the rectangle D' can be proved to vanish in the limit, we obtain the formula

$$(7)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\phi(lx+my)\,dx\,dy}{k+(ax+by)i}=\frac{2\pi\,\mathrm{sgn}\,\Delta}{am-bl}\Phi(lx_0+my_0).$$

But the particular case in which $\phi(u)=e^{ui}$ appears to be by far the most interesting.

§ 7. Since the double integral (2) is convergent, the two corresponding repeated integrals must also be convergent and have the same value as the double integral, provided only that the simple integrals with respect to x and y alone have a meaning. That this is so is readily verified. Suppose, for example, that $l>0$ and $\Delta>0$. Then

$$\int_{-\infty}^{\infty}\frac{e^{lxi}}{(k+byi)+axi}=\frac{2\pi}{a}e^{-l(k+byi)/a},$$

or zero, according as the imaginary part of $(ki-by)/a$ is positive or negative. This last condition reduces to

$$\alpha(\kappa-\beta'y)+\alpha'(\kappa'+\beta y)\gtrless 0,$$

or

$$\Delta y\lesseqgtr \alpha\kappa+\alpha'\kappa',$$

or

$$y\lesseqgtr y_0.$$

Hence

$$\int_{-\infty}^{\infty}e^{myi}dy\int_{-\infty}^{\infty}\frac{e^{lxi}dx}{k+(ax+by)i}=\frac{2\pi}{a}e^{-lk/a}\int_{-\infty}^{y_0}e^{\{m-(bl/a)\}yi}dy.$$

This integral is convergent if

$$R\left[\left(m-\frac{bl}{a}\right)i\right]>0,$$

or if $R[bi/a]<0$, or if $\Delta>0$, as is the case; and integration leads to the required value after a little reduction.

§ 8. We have so far left on one side the special case in which $\Delta=0$. In this case $\beta/\alpha=\beta'/\alpha'$ and the ratio b/a is real. We may therefore, without loss of generality, suppose that both a and b are real.

If x and y tend to ∞ in such a way that $ax+by$ remains finite, the subject of integration does not tend to zero. It is therefore natural to suppose (though it does not follow necessarily) that the integral is not convergent. In any case it is clear that the method of the preceding sections fails completely. In the first place the denominator vanishes if

$$\kappa=0, \quad \kappa'+ax+by=0,$$

i.e., not at all unless $\kappa=0$, and if $\kappa=0$ then *along a line* instead of at a point. And the argument which applied to the line integrals also fails completely.

That the integral is, in fact, not convergent may be seen as follows:—

Suppose, for definiteness, that κ and a are positive. Then

$$\int_{-\infty}^{\infty}\frac{e^{(lx+my)i}\,dx}{k+(ax+by)i}=\frac{2\pi}{a}\,e^{myi-\{(k+byi)l/a\}}$$

provided the imaginary part of $(ik-by)/a$ is positive, a condition which is certainly satisfied for all values of y. If the double integral were convergent the repeated integral obtained by integrating the right-hand side of the last equation would also be convergent, but

$$\int_{-\infty}^{\infty}e^{(am-bl)yi/a}\,dy$$

is not convergent, and therefore the double integral is not convergent.

§ 9. Similar considerations prevent the extension of the formulæ (5), (7) to multiple integrals of higher order. Consider, for example,

$$k+(ax+by+cz)i,$$

where $c=\gamma+i\gamma'$. We can always choose the ratios $x:y:z$ in such a way that x, y, and z tend to infinity, while

$$\alpha x+\beta y+\gamma z, \quad \alpha' x+\beta' y+\gamma' z$$

both remain finite; and the corresponding triple integral will not be convergent.

§ 10. It will perhaps be convenient if I summarize here the various classes of conditionally convergent multiple integrals that I have considered in this series of notes.

In Note XI. (vol. XXXII., p. 159) I evaluated the integrals

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{matrix}\cos\\ \sin\end{matrix} \{(a,\ b,\ c, f,\ g,\ h)(x,\ y,\ 1)^2\}\, dx\, dy,$$

where $ab-h^2>0$. Such previous instances as I had given (such as the integral

$$\int_0^{\infty}\int_0^{\infty} \sin(ax+by)\, x^{r-1}y^{s-1}\, dx\, dy$$

mentioned in Note X.) had been of a trivial character. In Note XVI. (vol. XXXIV., p. 6) I proved the convergence of the integral

$$\int_0^{\infty} e^{(c_1x_1+\ldots+c_nx_n)i}\, \phi(a_1x_1+\ldots+a_nx_n)\, dx_1\ldots dx_n,$$

where the c's and a's are all real, and every a is positive and no c zero, and $\phi(u)$ is a function of u which, together with its n^{th} derivative $\phi^{(n)}(u)$, tends continuously and steadily to zero. In Note XXI. (vol. XXXVII., p. 127) I found the actual value of the last integral in the case in which $\phi(u)=u^{-s}$.

The Frullanian double integrals, which I considered in Notes XX. and XXII. (vol. XXXVII., p. 96 and p. 154), are either absolutely convergent or convergent only in a much more special sense than that considered here. But my paper "On certain double integrals" (*Quarterly Journal*, vol. XXXVII., p. 360) contains a considerable number of conditionally convergent integrals, particular cases of the general class considered in Notes XVI. and XXI. Those considered in this note are somewhat similar in form but in reality of a quite different type.

A CLASS OF DETERMINANTS WHOSE ADJUGATE IS PERSYMMETRIC.

By *Thomas Muir, LL.D.*

1. ALTHOUGH the very first persymmetric determinant to make its appearance* was an adjugate, namely, the adjugate of Bezout's condensed eliminant of two equations of the n^{th} degree, no attempt seems to have been made to investigate the conditions whose fulfilment would in general ensure such a result. In some measure this is doubtless due to the fact that the question has not been separated from the subject of elimination. Thus the adjugate of the determinant

$$\begin{vmatrix} |a_0b_1| & |a_0b_2| & |a_0b_3| \\ |a_0b_2| & |a_0b_3|+|a_1b_2| & |a_1b_3| \\ |a_0b_3| & |a_1b_3| & |a_2b_3| \end{vmatrix}$$

has not been considered apart from the simultaneous existence of the equations

$$\left.\begin{aligned} a_0x^3+a_1x^2+a_2x+a_3=0 \\ b_0x^3+b_1x^2+b_2x+b_3=0 \end{aligned}\right\},$$

its persymmetry being usually established by showing that the elements of any row of the adjugate are proportional to $1, \xi, \xi^2$, where ξ is a root common to the two equations. It is not unusual also for those who employ this mode of proof to forget that the supposition of the existence of a common root entails the vanishing of the given determinant and therefore the vanishing of its adjugate.

2. Now the condition which must be fulfilled in order that the adjugate of

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma+\delta & \epsilon \\ \gamma & \epsilon & \zeta \end{vmatrix}$$

may be persymmetric is that the co-factor of the element in the $(1,3)^{\text{th}}$ place shall be equal to the co-factor of the element in the $(2,2)^{\text{th}}$ place; in other words, that

$$\beta\epsilon-\gamma^2-\gamma\delta=\alpha\zeta-\gamma^2$$

or

$$0=\alpha\zeta-\beta\epsilon+\gamma\delta,$$

* Jacobi, C. G. J., "De eliminatione...," *Crelle's Journ.*, XV. (1835), pp. 101–124.

a condition which is manifestly satisfied when $\alpha, \beta, \ldots, \zeta$ are, as in § 1, the two-line minors of the array

$$\begin{matrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3, \end{matrix}$$

but which, it is also manifest, may be satisfied otherwise.

Similarly there is no difficulty in finding that the adjugate of

$$\begin{vmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \gamma+\epsilon & \delta+\zeta & \eta \\ \gamma & \delta+\zeta & \eta+\theta & \iota \\ \delta & \eta & \iota & \kappa \end{vmatrix}$$

will be persymmetric if

$$\left.\begin{aligned} \alpha\theta - \beta\zeta + \gamma\varepsilon &= 0 \\ \alpha\iota - \beta\eta + \delta\varepsilon &= 0 \\ \alpha\kappa - \gamma\eta + \delta\zeta &= 0 \\ \beta\kappa - \gamma\iota + \delta\theta &= 0 \\ \varepsilon\kappa - \zeta\iota + \eta\theta &= 0 \end{aligned}\right\},$$

and we again see that these conditions* will be satisfied if $\alpha, \beta, \ldots, \kappa$ be taken equal to the two-line minors of the array

$$\begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4. \end{matrix}$$

3. The non-zero members of the conditional equations in § 1 and § 2 are pfaffians, being in § 2 the pfaffians

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \varepsilon & \zeta & \\ & \theta & \end{vmatrix}, \quad \begin{vmatrix} \alpha & \beta & \delta \\ & \varepsilon & \eta \\ & & \iota \end{vmatrix}, \quad \begin{vmatrix} \alpha & \gamma & \delta \\ & \zeta & \eta \\ & & \kappa \end{vmatrix},$$

$$\begin{vmatrix} \beta & \gamma & \delta \\ & \theta & \iota \\ & & \kappa \end{vmatrix}, \quad \begin{vmatrix} \varepsilon & \zeta & \eta \\ & \theta & \iota \\ & & \kappa \end{vmatrix},$$

* One of them is superfluous, the last, for example, being derivable from the first four by eliminating $\alpha, \beta, \gamma, \delta$.

that is to say, the principal minors of the quasi-pfaffian

$$\begin{vmatrix} \alpha & \beta & \gamma & \delta \\ & \varepsilon & \zeta & \eta \\ & & \theta & \iota \\ & & & \kappa \end{vmatrix}.$$

This observation enables us to formulate our results more conveniently, and a further help is got from using the umbrae 12, 13, ... instead of α, β, ... The next case, for example, is:—*If there be fifteen quantities* 12, 13, ..., 56, *such that all the four-line minors* of*

$$\begin{vmatrix} 12 & 13 & 14 & 15 & 16 \\ & 23 & 24 & 25 & 26 \\ & & 34 & 35 & 36 \\ & & & 45 & 46 \\ & & & & 56 \end{vmatrix}$$

vanish, then the adjugate of

$$\begin{vmatrix} 12 & 13 & 14 & 15 & 16 \\ 13 & 14+23 & 15+24 & 16+25 & 26 \\ 14 & 15+24 & 16+25+34 & 26+35 & 36 \\ 15 & 16+25 & 26+35 & 36+45 & 46 \\ 16 & 26 & 36 & 46 & 56 \end{vmatrix} \qquad \text{(I.)}$$

is persymmetric. The general theorem which includes the three is thus readily grasped, as well as the particular solution of the conditional equations which is suggested by the form of Bezout's eliminant.

* It may be well to recall that the pfaffian

$$\begin{vmatrix} 12 & 13 & 14 \\ & 23 & 24 \\ & & 34 \end{vmatrix}, \text{ i.e., } 12\,34 - 13\,.\,24 + 14\,.\,23,$$

has *four* frame-lines, namely,

$$12\ 13\ 14,\quad 12\ 23\ 24,\quad 13\ 23\ 34,\quad 14\ 24\ 34,$$

and is itself the minor obtained by deleting the 5th and 6th frame-lines of the six-line pfaffian specified in the enunciation. See my text-book on *Determinants*, § 162.

4. Our next point is that the said particular solution, when considered in connection with the Law of Extensible Minors,* suggests a solution that is much more comprehensive. This we may illustrate by returning to the second case of § 2, where it is seen that the substitution of

$$|a_0b_1|,\ |a_0b_2|,\ \ldots,\ |a_3b_4|$$

in place of $\alpha, \beta, \ldots, \kappa$ gives us a four-line determinant whose adjugate is persymmetric, the persymmetry being ensured by reason of the existence of the identities

$$\left.\begin{array}{l}
|a_0b_1|\,|a_2b_3| - |a_0b_2|\,|a_1b_3| + |a_0b_3|\,|a_1b_2| = 0\\
|a_0b_1|\,|a_2b_4| - |a_0b_2|\,|a_1b_4| + |a_0b_4|\,|a_1b_2| = 0\\
|a_0b_1|\,|a_3b_4| - |a_0b_3|\,|a_1b_4| + |a_0b_4|\,|a_1b_3| = 0\\
|a_0b_2|\,|a_3b_4| - |a_0b_3|\,|a_2b_4| + |a_0b_4|\,|a_2b_3| = 0\\
|a_1b_2|\,|a_3b_4| - |a_1b_3|\,|a_2b_4| + |a_1b_4|\,|a_2b_3| = 0
\end{array}\right\}.$$

Now, by the law just mentioned, these identities continue to hold if each determinant in them is "extended" by $c_5d_6\ldots$. This enables us at once to assert that the determinant

$$\begin{vmatrix}
|a_0b_1c_5d_6\ldots| & |a_0b_2c_5d_6\ldots| & |a_0b_3c_5d_6\ldots| & |a_0b_4c_5d_6\ldots|\\
|a_0b_2c_5d_6\ldots| & |a_0b_3c_5d_6\ldots| + |a_1b_2c_5d_6\ldots| & |a_0b_4c_5d_6\ldots| + |a_1b_3c_5d_6\ldots| & |a_1b_4c_5d_6\ldots|\\
|a_0b_3c_5d_6\ldots| & |a_0b_4c_5d_6\ldots| + |a_1b_3c_5d_6\ldots| & |a_1b_4c_5d_6\ldots| + |a_2b_3c_5d_6\ldots| & |a_2b_4c_5d_6\ldots|\\
|a_0b_4c_5d_6\ldots| & |a_1b_4c_5d_6\ldots| & |a_2b_4c_5d_6\ldots| & |a_3b_4c_5d_6\ldots|
\end{vmatrix}$$

has its adjugate persymmetric, and that generally *if each of the two-line minors of the array*

$$\begin{array}{ccccc} a_0 & a_1 & a_2 & \ldots & a_m\\ b_0 & b_1 & b_2 & \ldots & b_m \end{array}$$

be "extended" by the annexation of $c_pd_qe_r\ldots$, *where*

$$m < p < q < r < \ldots,$$

the $\frac{1}{2}m(m+1)$ *quantities thus obtained, namely,*

$$|a_0b_1c_pd_qe_r\ldots|,\ \ldots,\ |a_{m-1}b_mc_pd_qe_r\ldots|,$$

* See *Trans. R. S. Edinb.*, xxx. (1881), pp. 1–4.

when substituted for

$$|a_0b_1|, \ldots, |a_{m-1}b_m|$$

in Bezout's condensed eliminant of the m^{th} *order produce a determinant whose adjugate is persymmetric.* (II.)

5. In the third place the original particular solution, when considered in connection with the Law of Complementaries, gives rise to another of like character. For, returning to the set of five identities used in § 4 and substituting for each two-line determinant its complementary with respect to $|a_0b_1c_2d_3e_4|$, we have

$$\left.\begin{array}{l} |c_2d_3e_4|\,|c_0d_1e_4| - |c_1d_3e_4|\,|c_0d_2e_4| + |c_1d_2e_4|\,|c_0d_3e_4| = 0 \\ |c_2d_3e_4|\,|c_0d_1e_3| - |c_1d_2e_4|\,|c_0d_2e_3| + |c_1d_2e_3|\,|c_0d_3e_4| = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ |c_0d_3e_4|\,|c_0d_1e_2| - |c_0d_2e_4|\,|c_0d_1e_3| + |c_0d_2e_3|\,|c_0d_1e_4| = 0 \end{array}\right\}.$$

In other words the five conditional equations of §2 are satisfied by taking κ, ι, ..., β, α equal to the three-line minors of the array

$$\begin{array}{ccccc} c_0 & c_1 & c_2 & c_3 & c_4 \\ d_0 & d_1 & d_2 & d_2 & d_4 \\ e_0 & e_1 & e_2 & e_3 & e_4; \end{array}$$

and it thus follows that the adjugate of

$$\left|\begin{array}{cccc} |c_2d_3e_4| & |c_1d_3e_4| & |c_1d_2e_4| & |c_1d_2e_3| \\ |c_1d_3e_4| & |c_1d_2e_4| + |c_0d_3e_4| & |c_1d_2e_3| + |c_0d_2e_4| & |c_0d_2e_3| \\ |c_1d_2e_4| & |c_1d_2e_3| + |c_0d_2e_4| & |c_0d_2e_3| + |c_0d_1e_4| & |c_0d_1e_3| \\ |c_1d_2e_3| & |c_0d_2e_3| & |c_0d_1e_3| & |c_0d_1e_2| \end{array}\right|$$

is persymmetric, and generally that, *if in Bezout's condensed eliminant of the* m^{th} *order there be substituted for*

$$|a_0b_1|, |a_0b_2|, \ldots, |a_{m-1}b_m|$$

their complementary minors in any determinant of the $(m+1)^{\text{th}}$ *order having*

$$\begin{array}{ccccc} a_0 & a_1 & a_2 & \ldots & a_m \\ b_0 & b_1 & b_2 & \ldots & b_m \end{array}$$

for its first two rows, there is obtained a determinant whose adjugate is persymmetric. (III.)

6. Although the solution obtained in § 5 for the set of five conditional equations of § 2 is here derived from the solution given in the latter section, the one solution is in no sense subordinate to the other. Save for the accident of the connection between the first and the problem of elimination, the other might have made an equally early appearance in the pure theory of determinants. As a consequence the two results to which they lead may be formally enunciated together in such a way as to show their equality of rank. Thus. *if* 12, 13, ..., 45 *be the two-line minors of any two-by-five array, and* 123, 124, ..., 345 *be the three-line minors of any three-by-five array, then*

$$\begin{vmatrix} 12 & 13 & 14 & 15 \\ 13 & 14+23 & 15+24 & 25 \\ 14 & 15+24 & 25+34 & 35 \\ 15 & 25 & 35 & 45 \end{vmatrix}$$

and

$$\begin{vmatrix} 123 & 124 & 134 & 234 \\ 124 & 134+125 & 234+135 & 235 \\ 134 & 234+135 & 235+145 & 245 \\ 234 & 235 & 245 & 345 \end{vmatrix}$$

have their adjugates persymmetric. (IV.)

As a further consequence, the Law of Extensible Minors may be called into operation in connection with the second result exactly as it has been done in § 4 in connection with the first result, the outcome being that in the four-line determinant of § 5 we may, with impunity, "extend" $|c_2d_3e_4|$, $|c_1d_3e_4|$, ... by annexing f_5g_6... to each, and generally the formulation of a theorem coordinate with and analogous to (II.). (V.)

7. It is known that the elements of the adjugate of Bezout's eliminant of the m^{th} order are expressible as determinants of the $(2m-2)^{\text{th}}$ order: for example, the adjugate of the three-line determinant in § 1 is

$$\begin{vmatrix} \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \\ \mu_3 & \mu_4 & \mu_5 \end{vmatrix},$$

where μ_r is the minor obtained from the array

$$\begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & . \\ b_0 & b_1 & b_2 & b_3 & . \\ . & a_0 & a_1 & a_2 & a_3 \\ . & b_0 & b_1 & b_2 & b_3, \end{array}$$

or μ say, by deleting the r^{th} column and prefixing the sign $(-1)^{r-1}$. There is, however, one very interesting case in which the said elements are expressible as determinants of the same order as the eliminant itself, namely, the case where the b's are all equal to 1: for example, the adjugate of

$$\begin{vmatrix} a_1-a_2 & a_1-a_3 & a_1-a_4 & a_1-a_5 \\ a_1-a_3 & a_1-a_4+a_2-a_3 & a_1-a_5+a_2-a_4 & a_2-a_5 \\ a_1-a_4 & a_1-a_5+a_2-a_4 & a_2-a_5+a_3-a_4 & a_3-a_5 \\ a_1-a_5 & a_2-a_5 & a_3-a_5 & a_4-a_5 \end{vmatrix}$$

is

$$\begin{vmatrix} A_4 & A_5 & A_1 & A_2 \\ A_5 & A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 & A_4 \\ A_2 & A_3 & A_4 & A_5 \end{vmatrix}$$

or $$P(A_4,\ A_5,\ A_1,\ A_2,\ A_3,\ A_4,\ A_5),$$

where $(-1)^{r-1}A_r$ stands for the r^{th} four-line minor of the

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_1 \\ a_3 & a_4 & a_5 & a_1 & a_2 \\ 1 & 1 & 1 & 1 & 1. \end{array} \qquad \text{(VI.)}$$

We may note further that, since

$$A_1+A_2+A_3+A_4+A_5=0,$$

it is easy to show that

$$\begin{aligned} P(A_4,\ A_5,\ A_1,\ A_2,\ A_3,\ A_4,\ A_5) &= P(A_5,\ A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ A_1) \\ &= P(A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ A_1,\ A_2) \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

8. The determinant taken as a special example in § 7 has three other forms, namely,

$$\begin{vmatrix} a_1-a_2 & a_2-a_3 & a_3-a_4 & a_4-a_5 \\ a_1-a_3 & a_2-a_4 & a_3-a_5 & a_4-a_1 \\ a_1-a_4 & a_2-a_5 & a_3-a_1 & a_4-a_2 \\ a_1-a_5 & a_2-a_1 & a_3-a_2 & a_4-a_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_1 \\ a_3 & a_4 & a_5 & a_1 & a_2 \\ a_4 & a_5 & a_1 & a_2 & a_3 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix},$$

and

$$\begin{vmatrix} a_1-a_2 & a_2-a_3 & a_3-a_4 & a_4-a_5 \\ a_2-a_3 & a_3-a_4 & a_4-a_5 & a_5-a_1 \\ a_3-a_4 & a_4-a_5 & a_5-a_1 & a_1-a_2 \\ a_4-a_5 & a_5-a_1 & a_1-a_2 & a_2-a_3 \end{vmatrix}.$$

The last is especially interesting in that it is persymmetric and yet has not a persymmetric adjugate; while its equivalent, the original determinant, which is only axisymmetric, does have its adjugate persymmetric.

In this connection it may be noted that the condition for persymmetry in the adjugate of

$$\begin{vmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{vmatrix}$$

is

$$\begin{vmatrix} a & b & c & d & e \\ b & c & d & e & f \\ c & d & e & f & g \end{vmatrix} = 0;$$

and generally it is the case that *the persymmetric determinant of the n^{th} order will have its adjugate also persymmetric, if the principal minors of the corresponding persymmetric array of $n-1$ rows and $n+1$ columns all vanish.* (VII.)

Further, it is noteworthy that, although the number of different principal minors of the square persymmetric array of the n^{th} order is the same as the number of principal minors of the corresponding persymmetric array of $n-1$ rows and $n+1$ columns, being in the one case

$$n+(n-1)+\ldots+2+1, \text{ } i.e., \text{ } \tfrac{1}{2}n(n+1),$$

and in the other

$$C_{n+1,n-1}, \quad i.e., \; C_{n+1,2}, \quad i.e., \; \tfrac{1}{2}n(n+1),$$

the two series of minors are not identical. Thus, taking the case where $n=4$, the different principal minors of the four-by-four array are

123	123	123	123
123	124	134	234
	124	124	124
	124	134	234
		134	134
		134	234
			234
			234,

where $\begin{matrix} rst \\ \rho\sigma\tau \end{matrix}$ is used to denote the minor which occupies the r^{th}, s^{th}, t^{th} rows and ρ^{th}, σ^{th}, τ^{th} columns of the array; and the principal minors of the three-by-five array are 123, 124, ..., 345, where hkl stands for the minor whose columns are the h^{th}, k^{th}, l^{th} columns of the array. Three of the former, namely,

124	124	124
124'	134'	134

are not found among the latter; and three of the latter, namely,

125, 135, 145

are not found among the former. Nevertheless if to each of the second three there be added another member of the same series, the sums are respectively equal to the first three, namely,*

$$\begin{matrix} 124 \\ 123 \end{matrix} = 125 + 134, \quad \begin{matrix} 124 \\ 134 \end{matrix} = 135 + 234, \quad \begin{matrix} 134 \\ 134 \end{matrix} = 145 + 235.$$

From this it follows that the adjugate of

$$\begin{vmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{vmatrix}$$

* For these and the corresponding identities in the example immediately following, see *Trans. R. Soc. Edinburgh*, XL. (1902), pp. 511–533

can be expressed in terms of the principal minors of the array

$$\begin{matrix} a & b & c & d & e \\ b & c & d & e & f \\ c & d & e & f & g \end{matrix}$$

being, in fact,

$$\begin{vmatrix} 345 & -245 & 235 & -234 \\ -245 & 235+145 & -234-135 & 134 \\ 235 & -234-135 & 134+125 & -124 \\ -234 & 134 & -124 & 123 \end{vmatrix},$$

which is manifestly not altered by the deletion of the minus signs. A glance, too, suffices to show that it would be persymmetric if

$$145=0, \quad 135=0, \quad 125=0,$$

a result in agreement with (VII.) above. Similarly, the adjugate of

$$P(a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i)$$

is equal to

$$\begin{vmatrix} 1234 & 1235 & 1245 & 1345 & 2345 \\ 1235 & 1245+1236 & 1345+1246 & 2345+1346 & 2346 \\ 1245 & 1345+1246 & 2345+1346+1256 & 2346+1356 & 2356 \\ 1345 & 2345+1346 & 2346+1356 & 2356+1456 & 2456 \\ 2345 & 2346 & 2356 & 2456 & 3456 \end{vmatrix},$$

where $hklm$ stands for the minor whose columns are the h^{th}, k^{th}, l^{th}, m^{th} columns of the array

$$\begin{matrix} a & b & c & d & e & f \\ b & c & d & e & f & g \\ c & d & e & f & g & h \\ d & e & f & g & h & i, \end{matrix} \qquad \text{(VIII.)},$$

and where the conditions for persymmetry are the vanishing of

$$1236, \quad 1246, \quad 1346, \quad 1256, \quad 1356, \quad 1456.$$

9. Looking to the adjugate of $P(a, b, c, d, e, f, g)$, given in § 8 and to the condition for persymmetry of *its* adjugate given in § 3, we see that the adjugate of the adjugate of

$P(a, b, c, d, e, f, g)$ will be persymmetric if all the four-line minors of the quasi-pfaffian

$$\begin{vmatrix} 345 & 245 & 235 & 234 \\ & 145 & 135 & 134 \\ & & 125 & 124 \\ & & & 123 \end{vmatrix}$$

vanish. But that these do vanish is known from the second sentence of § 5. Consequently *the adjugate of the adjugate of a persymmetric determinant is persymmetric.* (IX.)

This is of no special importance save as a verification, being evident from considering what the elements of the adjugate of the adjugate are in general. For example, if Δ stand for $|a_1 b_2 c_3|$, we have the adjugate of adjugate of

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1\Delta & a_2\Delta & a_3\Delta \\ b_1\Delta & b_2\Delta & b_2\Delta \\ c_1\Delta & c_2\Delta & c_3\Delta \end{vmatrix}.$$

10. The known theorem referred to in §7 has an analogue connected with the determinant of (II.). Thus the adjugate of

$$\begin{vmatrix} |a_0 b_1 c_4| & |a_0 b_2 c_4| & |a_0 b_3 c_4| \\ |a_0 b_2 c_4| & |a_0 b_3 c_4| + |a_1 b_2 c_4| & |a_1 b_3 c_4| \\ |a_0 b_3 c_4| & |a_1 b_3 c_4| & |a_2 b_3 c_4| \end{vmatrix},$$

where the "extension" is c_4, is

$$\begin{vmatrix} \nu_1 & \nu_2 & \nu_3 \\ \nu_2 & \nu_3 & \nu_4 \\ \nu_3 & \nu_4 & \nu_5 \end{vmatrix},$$

where $(-1)^{r-1}\nu_r$ is the minor obtained by deleting the r^{th} column of the array

$$\begin{array}{ccccccc} a_0 & a_1 & a_2 & a_3 & . & a_1 & . \\ b_0 & b_1 & b_2 & b_3 & . & b_4 & . \\ c_0 & c_1 & c_2 & c_3 & . & c_4 & . \\ . & a_0 & a_1 & a_2 & a_3 & . & a_4 \\ . & b_0 & b_1 & b_2 & b_3 & . & b_4 \\ . & c_0 & c_1 & c_2 & c_3 & . & c_4 \end{array} \qquad (X.).$$

Cape Town, S.A.,
May 29th, 1910.

A SYSTEM OF CIRCLES DERIVED FROM A CUBIC SPACE CURVE AND THE PROPERTIES OF A CERTAIN CONFIGURATION OF FIFTEEN LINES.

By *H. Bateman.*

§ 1. It is well known that the geometry of a singly infinite system of circles in a plane may be derived from that of a curve in space by projecting stereographically* the circles cut out on an arbitrary sphere by the osculating planes of the curve. This correspondence may be discussed in detail with the help of the curve which is the locus of the poles of the osculating planes with regard to the sphere. By a theorem due to Hachette the pole of a plane projects into the centre of the circle which is the projection of the circle cut out on the sphere by the plane, hence the curve just mentioned projects into the curve which is the locus of the centres of the system of circles.

The method gives a correspondence between a circle in a plane and the plane which cuts out the corresponding circle on the sphere. If two circles in the plane cut orthogonally, the corresponding planes are conjugate with regard to the sphere.

The value of the method may be illustrated by transforming the theorem that the angle in a semi-circle is a right angle. The corresponding theorem is as follows:

If a tetrahedron is inscribed in a sphere (or quadric) and two faces are conjugate with regard to the sphere the other two faces are also conjugate.

Returning to the study of the two curves in space and the corresponding system of circles, we see that the points on the original curve project into points which are the centres of

* The method of stereographic projection appears to have originated with Hipparchus (2 A.D.) and Ptolemy. The name stereographic projection was introduced by Aiguillon in 1613. The fact that the transformation is a conformal one was realised by Mercator (1587). The method was developed and used in geometry by Hachette, Poncelet, and Chasles. See Chasles' *Aperçu historique* (1837); also papers by Dandelin, *Gergonne Annales*, vol. xvi. (1825–26), pp. 322–27; *Bruxelles Mémoires*, vol. iv. (1827), pp. 13–47: and by Chasles, *Gergonne Annales*, vol. xviii. (1827–28), pp. 305–320; vol. xix. (1828–29), pp. 157–175.

The following references may be found useful:—F. Klein, *Math. Ann.*, Bd. 5 (1871); Fricke u Klein, *Die Theorie der Automorphen Functionen*, Bd. 1; Darboux, *Leçons sur les systèmes orthogonoux*, p. 169; Holzmüller, *Elemente du Stereometrie*, p. 250.

circles cutting orthogonally three consecutive circles of the system. These circles may be regarded as the osculating orthogonal circles of the system; they form a singly infinite system which may be called the reciprocal of the first. It is clear that the relation between the two systems of circles is a reciprocal one.

If we denote the two systems of circles by S and S', and the curves which are the loci of their centres by C and C', we easily see that the following relations hold:—

(1) The lines belonging to S are inflexional tangents of C', and conversely the lines belonging to S' are inflexional tangents of C.

(2) The radical axis of two consecutive members of S is a tangent to C', and the radical axis of two consecutive members of S' is a tangent to C.

(3) The envelope of the circle S is the locus of the limiting points of consecutive members of the system of circles S'. The point circles of the system S' are cusps on the envelope.

§ 2. In the case of the system of circles derived from a cubic curve in space the two systems S and S' are of the same type, and the curves C and C' are both rational cubics.

The *cubic system* of circles S may be specified by an equation of the form

$$A(x^2+y^2)+2Gx+2Fy+B=0,$$

where A, G, F, B are cubic functions of a parameter t.

It is easy to see that there are three circles through an arbitrary point and three circles orthogonal to an arbitrary circle. There are three lines belonging to the system and six point circles, these are given respectively by the values of t, for which

$$A=0 \text{ and } G^2+F^2-AB=0.$$

The radical axis of a given circle and a variable circle of the system envelops a tricusped quartic, for the coefficients in its equation are cubic functions of t. The cusps are at the centres of the three circles of the system S', which are orthogonal to the given circle, for three consecutive radical axes must pass through the centre of one of those circles. The tricusped quartic evidently touches the three lines belonging to the system S; hence it is related to the curve C' in such a way that its cusps lie on C' and it touches the three inflexional tangents of C'.

In the particular case when the given circle is a point O, the tricusped quartic touches the six lines which bisect at right angles OA_1, OA_2, OA_3, OA_4, OA_5, OA_6, where $A_1, \ldots, A_6$ are the point circles of the system S. It appears then that the six points and the three lines of the system S are related in such a way that, if O is any point, the six lines bisecting OA_1, OA_2, ..., OA_6 at right angles and the three lines of the system envelop a quartic with three cusps.

In the particular case when the given circle belongs to the system S, the tricusped quartic reduces to a conic. This conic touches the three lines of the system S and also touches the curve C' at the point where C' is touched by the radical axis of the given circle and a consecutive circle.

If the given circle coincides with the point circle A_1, the conic is the one which touches the lines bisecting A_1A_2, A_1A_3, A_1A_4, A_1A_5, A_1A_6 at right angles. We thus have a conic associated with each of the points A_r, and these six conics have three common tangents, viz., the three lines of the system or the three inflexional tangents of the curve C'.

We thus have a method of constructing the three lines of the system S when the six-point circles are given. It appears that the system of circles is completely and uniquely determined by these six points.* To see this we have only to project back on to the sphere and remark that there is only one twisted cubic through the corresponding points on the sphere.

The six-point circles of a system S may be taken to be any six arbitrary points, but the system S degenerates when three or more lie on a line. The geometry of the system of circles is thus connected with the geometry of six arbitrary points in a plane.

It should be remarked that if we invert from a point O, the system S inverts into another system s of the same character, the curve c, which is the locus of the centres of circles belonging to the system s, is the reciprocal of the tricusped quartic enveloping the polars of O with regard to the circles of S. The nature of the envelope may be deduced from the fact that c is a nodal cubic.

An interesting property of the systems of circles S and S' may be derived from the theorem that there are three osculating planes of a twisted cubic which pass through a given point O and their points of contact lie on a plane through O. Projecting stereographically we find that there are three

* It is also completely and uniquely determined by six circles of the system

circles of the system S' which are orthogonal to a given circle A. If L, M, N are the three circles of the system S, at which it is osculated by the three circles of the system S', these circles are cut orthogonally by a circle A' which is orthogonal to A. A one-one correspondence of a reciprocal nature is thus established between circles A and A' in a plane with the aid of the two systems S and S'. A circle of the system S is transformed in this way into a circle of the system S'.

If we take the triangle formed by the three lines of the system S as the fundamental triangle of a quadratic transformation between lines in the plane, the conics which touch the sides of the triangle correspond to the points; and a nodal cubic which has the sides of the triangle as inflexional tangent corresponds to a conic circumscribing the triangle. It follows from this that the system of conics which are inscribed in the triangle and touch the nodal cubic C' are transformed into the points of a conic circumscribing the triangle.

Since there is a conic associated with each circle of the system S and the radical axis of two circles is a common tangent to the two associated conics, it follows that the radical axis is transformed into the line joining the two points corresponding to the two conics.

If we take any six circles of the system S we shall have six associated conics, and these are transformed into six points on the conic circumscribing the fundamental triangle. The radical axes of the circles taken in pairs correspond with the lines joining the six points in pairs. Since a cubic system of circles is in general defined uniquely when six members of it are given, we have the following theorem:

The radical axes of six circles taken in pairs may be transformed by means of a quadratic transformation of the lines in a plane into the fifteen lines joining six points on a conic. The fundamental triangle of the transformation is formed by the three lines belonging to the cubic system of circles determined by the six given ones.

In the particular case when the six circles reduce to points, the radical axes become the lines bisecting at right angles the distances between the points.

It is clear that various properties of the configuration, formed out of six arbitrary circles or six arbitrary points in a plane, may be derived from the properties of the figure formed out of six arbitrary points on a conic, for instance, we may obtain an analogue of Pascal's theorem.

The geometry of the system of lines may be studied analytically by taking as triangle of reference the triangle formed by the three lines belonging to the cubic system

of circles. The equation of a conic circumscribing this triangle is

$$\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the equation of the line joining two points (x_1, y_1, z_1), (x_2, y_2, z_2) on this conic is

$$\frac{fx}{x_1x_2}+\frac{gy}{y_1y_2}+\frac{hz}{z_1z_2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Let the quadratic transformation be such that the line

$$lx+my+nz=0$$

is transformed into

$$\frac{ax}{l}+\frac{by}{m}+\frac{cz}{n}=0.$$

Then the equation of the radical axis of the circles (1) and (2) is

$$\frac{ax_1x_2x}{f}+\frac{by_1y_2y}{g}+\frac{cz_1z_2z}{h}=0 \quad \ldots\ldots\ldots\ldots (3),$$

this being the line which corresponds to (2) in the quadratic transformation between the lines of the plane.

The radical axes of the circles (1)(2), (2)(3), (3)(1) evidently meet in the point whose coordinates are

$$\frac{f^2}{ax_1x_2x_3}, \quad \frac{g^2}{by_1y_2y_3}, \quad \frac{h^2}{cz_1z_2z_3},$$

for the three equations of type (3) are satisfied on account of the equations

$$\frac{f}{x_3}+\frac{g}{y_3}+\frac{h}{z_3}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1').$$

This point is the radical centre of the circles (1)(2)(3). In the same way the coordinates of the radical centre of (4)(5)(6) are found to be proportional to

$$\frac{f^2}{ax_4x_5x_6}, \quad \frac{g^2}{by_4y_5y_6}, \quad \frac{h^2}{cz_4z_5z_6}.$$

Now put

$$\frac{f^4}{a^2x_1x_2x_3x_4x_5x_6}=\lambda,$$

$$\frac{g^4}{b^2y_1y_2y_3y_4y_5y_6}=\mu,$$

$$\frac{h^4}{c^2z_1z_2z_3z_4z_5z_6}=\nu,$$

then we see that if (α, β, γ), $(\alpha', \beta', \gamma')$ are the coordinates of these two radical centres, we have the relations

$$\frac{\alpha\alpha'}{\lambda} = \frac{\beta\beta'}{\mu} = \frac{\gamma\gamma'}{\nu}.$$

Hence the two radical centres are corresponding points in a quadratic transformation determined by the quantities λ, μ, ν. This quadratic transformation is the same however we group the six circles into triads; hence the twenty radical centres can be divided into ten pairs, which are corresponding points in a quadratic transformation, the base triangle of which is the triangle formed by the three lines belonging to the cubic system of circles containing the given six.

Let the radical centre of the circles (1, 2, 3) be denoted by (123), and the radical axis of (1) and (2) by the symbol (12), then it is clear that the four points (123), (124), (125), (126) lie on a straight line. It follows then that the corresponding points (456), (356), (346), (345) lie on a conic (3456) circumscribing the base triangle, this being the conic corresponding to the line (12).

To obtain a further property of this conic we make use of the theorem that, if (PP'), (QQ') are two pairs of corresponding points, then the points of intersection of $(PQ, P'Q')$, $(PQ', P'Q)$ form another pair of corresponding points. [This follows from the fact that two corresponding points are conjugate points with regard to all conics through the points

$$\pm\sqrt{\lambda}, \quad \pm\sqrt{\mu}, \quad \pm\sqrt{\nu},$$

and from Hesse's theorem that, if two pairs of opposite corners of a quadrilateral are conjugate points with regard to a conic, the third pair are also conjugate.]

Let us apply this theorem to the four points (123), (456), (124), (356). It is clear that the line joining (123), (124) is the line (12), and that the line joining (456), (356) is the line (56). The point corresponding to their point of intersection should therefore lie on the conics (3456), (1234). Hence the point of intersection of (123), (356) and (124), (456) lies on both these conics. Denoting this point by the symbol (12, 56), we have the result that the conic (3456) passes through the following points:

(345), (456), (563), (634), (12, 34), (12, 45),

(12, 53), (12, 36), (12, 46), (12, 56).

Thus we have a system of fifteen conics, each of which passes through ten points of the figure, and these conics all have three common points.

§ 3. The cubic system of circles may be generated in another way by taking two circular cubics which intersect in a given point O. We may establish a one-one correspondence between chords OPP', OQQ' of the two cubics by choosing them so that

$$OP.OP' = OQ.OQ',$$

then $PP'QQ'$ lie on a circle of the system.

There are two positions of the chord OPP' in which it is in the same straight line as OQQ', the two lines thus obtained are lines belonging to the system. The other line belonging to the system is obtained by drawing chords $OR\infty$, $OS\infty'$ parallel, respectively, to the asymptotes of the two cubics, and finding the other points R and S in which they meet the corresponding cubic. The line RS is then the third line of the system.

If the circle $PP'QQ'$ cuts one cubic again in UU' and the other cubic in VV', it appears that UU' always passes through R and VV' always passes through S.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy*, Trinity College, Cambridge.

XXIX.

Two convergence theorems.

§ 1. The two theorems stated below are of some interest in themselves and of fairly frequent use in such theories as that of summable integrals. They are suggested by a theorem concerning series proved by Dr. Bromwich.*

§ 2. Theorem 1. *Suppose* (i) *that ϕ is a function of x which has a continuous derivative*, (ii) *that ϕ is positive and tends steadily to ∞ with x*, (iii) *that f is a function of x which possesses a derivative integrable and absolutely integrable in*

* *Math. Annalen*, Bd. 65, p. 361.

*any finite interval,** (iv) *that* $f \to 0$ *as* $x \to \infty$, (v) *that the integral*

$$\int_a^\infty \phi(x)\,|f'(x)|\,dx \tag{1}$$

is convergent. Then the integral

$$\int_a^\infty \phi'(x)\,|f(x)|\,dx \tag{2}$$

will be convergent and less than the integral (1).

Let
$$F(x) = \int_x^\infty |f'(t)|\,dt;$$

the convergence of this integral follows from conditions (v) and (ii). Then

$$|f(x)| = \left|\int_x^\infty f'(t)\,dt\right| \leq F(x).$$

Again

$$\int_a^X \phi'(x)\,F(x)\,dx = \phi(X)\,F(X) - \phi(a)\,F(a) - \int_a^X \phi(x)\,F'(x)\,dx.$$

In this equation every term is positive save the term $-\phi(a)F(a)$, which is negative. Also

$$F'(x) = -|f'(x)|.$$

Hence

$$\int_a^X \phi'(x)\,F(x)\,dx < \phi(X)\int_X^\infty |f'(x)|\,dx + \int_a^X \phi(x)\,|f'(x)|\,dx$$
$$< \int_a^\infty \phi(x)\,|f'(x)|\,dx,$$

which proves the theorem. More precisely

$$\int_a^\infty \phi'(x)\,|f(x)|\,dx \leq \int_a^\infty \phi(x)\,|f'(x)|\,dx - \phi(a)\,F(a); \tag{3}$$

and *a fortiori*

$$\int_a^\infty \phi'(x)\,|f(x)|\,dx \leq \int_a^\infty \phi(x)\,|f'(x)|\,dx - \phi(a)\,|f(a)|. \tag{4}$$

* To say that f is integrable and absolutely integrable means that f and $|f|$ are integrable. In any interval in which f is limited, the integrability of f involves that of $|f|$, and the converse is not true. But, when f is unlimited, f may be integrable without $|f|$ being so.

§ 3. THEOREM 2. *Suppose* (i) *that* ϕ *is positive and tends steadily to* 0 *as* $x \to \infty$, (ii) *that the conditions* (i), (iii), *and* (v) *of Theorem* 1 *are satisfied. Then the integral* (2) *is convergent, and*

$$(5) \quad \int_a^\infty |\phi'(x)|\,|f(x)|\,dx \leq \int_a^\infty \phi(x)\,|f'(x)|\,dx + \phi(a)\,|f(a)|.$$

Let $$F(x) = \int_a^x |f'(t)|\,dt.$$

Then

$$|f(x)| = \left|f(a) + \int_a^x f'(t)\,dt\right| \leq |f(a)| + F(x),$$

$$\int_a^X \phi'(x) F(x)\,dx = \phi(X) F(X) - \int_a^X \phi(x) F'(x)\,dx.$$

In the last equation $\phi(X) F(X)$ is positive and the other two terms negative. Hence

$$0 < -\int_a^X \phi'(x) F(x)\,dx < \int_a^X \phi(x)\,|f'(x)|\,dx.$$

And $$\int_a^X \phi'(x)\,|f(x)|\,dx$$

is negative and numerically not greater than

$$-\int_a^X \phi'(x) F(x)\,dx - |f(a)| \int_a^X \phi'(x)\,dx$$

$$< \int_a^X \phi(x)\,|f'(x)|\,dx + \phi(a)\,|f(a)|;$$

whence the result of the theorem follows.

§ 4. The following examples show that the sign of equality is possible in the inequalities (3), (4), (5).

Let $\phi = x^r$, $f = x^{-s}$, where $0 < r < s$. Then the conditions of Theorem 1 are satisfied if $a > 0$. Also

$$\int_a^\infty \phi f'\,dx = -\frac{sa^{r-s}}{s-r}, \qquad \int_a^\infty \phi' f\,dx = \frac{ra^{r-s}}{s-r},$$

$$\phi(a)\,|f(a)| = \phi(a) \int_a^\infty |f'|\,dx = a^{r-s},$$

and (3) and (4) hold with the sign of equality. Similarly, if $\phi = x^{-r}$, $f = x^s$, where $0 < s < r$, we see that the conditions of Theorem 2 are satisfied, and that (5) holds with the sign of equality.

§ 5. As an example of the use of Theorem 1 suppose that we write

$$\phi = x^r, \quad f = \frac{d^r \psi}{dx^r}$$

where $\psi \to 0$. Then it is an immediate deduction from the theorem that the convergence of

$$\int_a^\infty x^r \left| \frac{d^{r+1}\psi}{dx^{r+1}} \right| dx$$

involves that of all the integrals

$$\int_a^\infty x^{r-s} \left| \frac{d^{r-s+1}\psi}{dx^{r-s+1}} \right| dx$$

for $s = 1, 2, \ldots, r$. The conditions stated also imply that

$$\psi' \to 0, \ \psi'' \to 0, \ \ldots, \ \psi^{(r)} \to 0.$$

For

$$\int^\infty \psi^{(r+1)} dx$$

is convergent, and so $\psi^{(r)}$ tends to a limit as $x \to \infty$, and this limit can only be zero. Similarly $\psi^{(r-1)} \to 0$, and so on.

As an example of Theorem 2 let us suppose

$$\phi(x) = 1/x.$$

Then we see that the convergence of

$$\int^\infty \frac{|f'(x)|}{x} dx \tag{6}$$

involves that of

$$\int^\infty \frac{|f(x)|}{x^2} dx, \tag{7}$$

and so also that of

$$\int_X^\infty \left| \frac{d}{dx} \frac{f(x)}{x} \right| dx. \tag{8}$$

Pringsheim* has given, as one of a set of sufficient conditions for the relations

$$\int_a^\infty \frac{f(x)}{x} \begin{matrix}\cos\\ \sin\end{matrix} \lambda x \, dx \to 0$$

(as $\lambda \to \infty$), the convergence of (8) or of both (6) and (7). It appears from what precedes that this condition may be simplified by demanding only the convergence of (6).

If we write $$f(x) = xg(x),$$

we see that the absolute convergence of

$$\int^\infty \left\{\frac{g(x)}{x} + g'(x)\right\} dx$$

involves that of

$$\int^\infty \frac{g(x)}{x}\, dx, \quad \int^\infty g'(x)\, dx, \tag{9}$$

and so also $g(x) \to 0$. It is worth remarking that the absolute convergence of the second integral (9) does not involve that of the first. For example, if

$$g(x) = e^{ai \log\log x} / \log x,$$

$$|g'(x)| < \frac{K}{x (\log x)^2},$$

and the second integral (9) is absolutely convergent. But

$$\int^\infty \frac{e^{ai\log\log x}}{x \log x}\, dx$$

is not convergent at all.

* *Math. Annalen*, Bd. 68, S. 384.

THE PRODUCTS OF q-HYPERGEOMETRIC FUNCTIONS.

By *Rev. F. H. Jackson, M.A.*

Introduction.

The expansion in power series, of products of hypergeometric functions, has been discussed by various writers,* notably by Goursat, Orr, Nicholson, and Watson. It has been proved by Appell that if the general solution of the equation

$$\frac{d^2y}{dx^2} = f_1(x)\frac{dy}{dx} + f_2(x)\,y \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be

$$y = c_1 y_1 + c_2 y_2,$$

then the equation, of which the general solution is

$$z = Ay_1^{\,2} + By_1y_2 + Cy_2^{\,2},$$

is

$$\frac{d^3z}{dx^3} - 3f_1(x)\frac{d^2z}{dx^2} + \left[2\{f_1(x)\}^2 - \frac{d}{dx}f_1(x) - 4f_2(x)\right]\frac{dz}{dx} + \left[4f_1(x)\,f_2(x) - 2\frac{d}{dx}f_2(x)\right]z = 0 \ldots (2).$$

In the special case, when (1) is the differential equation of the hypergeometric series with three elements, it can be shown that (2) reduces to a differential equation of a hypergeometric series with five elements provided that certain relations exist between the elements α, β, γ. Thus Goursat shows that if (1) be the differential equation

$$x(\vartheta+\alpha)(\vartheta+\beta)\,y - \vartheta(\vartheta+\gamma-1)\,y = 0 \ldots\ldots (3),$$

then (2) becomes

$$x(\vartheta+2\alpha)(\vartheta+2\beta)(\vartheta+\alpha+\beta)\,z - \vartheta(\vartheta+\gamma-1)(\vartheta+2\gamma-2)\,z = 0 \ldots\ldots (4)$$

provided $2\alpha+2\beta+1=2\gamma$. It is at once obvious that certain interesting results regarding the squares and products of pairs of hypergeometric series can be deduced from this theorem.

* Goursat, *Ann. de l'Ecole Normale Supérieure*, Ser 2, Tom. XII.; Orr, *Camb. Phil. Trans.*; Nicholson, *Phil. Mag.* (1905); Watson, *Quar. Jour. Math.* (1907); Appell, *Comptes Rendus*, Tom. XCI.

In the following work certain results for the q-series $F([\alpha][\beta][\gamma], x)$ will be obtained. It is remarkable that a q-differential equation, analogous to Appell's somewhat complex equation (2), should take the comparatively simple form

$$\{q^{4\vartheta}+f_1(qx)\,q^{3\vartheta}-f_1(x)f_2(qx)\,q^{\vartheta}-f_2(x)f_2(qx)\}\,z=0\ldots(5),$$

in which, as usual, ϑ denotes the symbolic operator $x\dfrac{d}{dx}$. Appell's equation (2), quoted above, can be deduced from (5) as a special case ($\lim q\to 1$). I shall show that (5) is satisfied by a product

$$\Phi(x)\,\Phi(qx)$$

provided that $\Phi(x)$ is any solution of

$$\{q^{4\vartheta}+f_1(x)\,q^{2\vartheta}+f_2(x)\}=0\ldots\ldots\ldots\ldots\ldots(6).$$

Of course $f_1(x)$ and $f_2(x)$ are supposed known functions. In case (6) is satisfied by a q-hypergeometric series $F([\alpha][\beta][\gamma], x)$, we shall see that $f_1(x)$, $f_2(x)$ are simple functions of the type $(ax+b)/(cx+d)$. Just as in the case of Goursat's deduction (4) from Appell's equation, so here, we shall deduce that (6) will take the form

$$x[2\vartheta+2\alpha][2\vartheta+2\beta]y-[2\vartheta][2\vartheta+2\gamma-2]y=0\ldots(7),$$

and (5) will reduce to

$$x[\vartheta+2\alpha][\vartheta+2\beta][2\vartheta+2\alpha+2\beta]z$$
$$-[\vartheta][2\vartheta+2\gamma-2][\vartheta+2\gamma-2]z=0\ldots(8),$$

provided $2\gamma=2\alpha+2\beta+1$; and, as usual, in the notation of my papers on q-functions, $[\vartheta]$ denotes

$$\frac{q^{\vartheta}-1}{q-1}.$$

Before dealing with equations (5) and (6), it will make the subsequent work easier to follow if we solve a special equation.

I.

The differential equation of q-hypergeometric series with three elements.

In dealing with q-series it is convenient to denote

$$\frac{q^{\alpha}-1}{q-1}\ \text{by}\ [\alpha],$$

and, in accordance with this notation, $[\vartheta]$ denotes a differential operative symbol

$$\frac{q^{\vartheta}-1}{q-1}, \text{ in which } \vartheta = x\frac{d}{dx}.$$

Now q^{ϑ} operating on x^n gives

$$q^n x^n,$$

so that $[\vartheta+\alpha]$ operating on x^n gives

$$[n+\alpha]\,x^n,$$

in analogy with the operator $(\vartheta+\alpha)$ giving

$$(\vartheta+\alpha)\,x^n = (n+\alpha)\,x^n.$$

Consider now a q-differential equation

$$\{x[\vartheta+\alpha][\vartheta+\beta]-[\vartheta][\vartheta+\gamma-1]\}\,y=0.$$

Substitute $\Sigma\alpha_r x^{\theta+r}$ for y in the differential equation. On performing the operations indicated, we obtain an indicial equation

$$[\theta][\theta+\gamma-1]=0.$$

I need not here enter into a detailed solution of the equation which has been discussed by me in the *American Journal of Mathematics*, October, 1910.

The indicial equation has two principal roots,

$$0, \quad 1-\gamma,$$

and besides these a doubly infinite set of roots corresponding to each of these principal roots and given in general by

$$\left.\begin{aligned}\theta &= 0+\frac{2r\pi i}{\log q+2s\pi i},\\ &= 1-\gamma+\frac{2r\pi i}{\log q+2s\pi i},\end{aligned}\right\}\ (r, s=0, 1, 2, 3, \ldots).$$

Of course the differential equation is of infinite order, and the general solution must embrace an infinite number of particular solutions, arising respectively from each of the roots of the indicial equation. We can, however, group the particular solutions in pairs, and the solution corresponding to

the principal values of θ, I term a principal solution. Such solution is

$$y = c_1 F([\alpha][\beta][\gamma], x) + c_2 x^{1-\gamma} F([\alpha-\gamma+1][\beta-\gamma+1][2-\gamma], x).$$

For certain critical values of α, β, γ, we may fail to obtain solutions in this form, but in such cases we can obtain solutions* involving $\log x$, by a method analogous to the method of Frobenius.

If we replace q by q^2 in the preceeding work, we shall have the differential equation

$$\{x[2\vartheta+2\alpha][2\vartheta+2\beta]-[2\vartheta][2\vartheta+2\gamma-2]\}\, y = 0 \ldots (9)$$

with a principal solution

$$c_1 F([2\alpha][2\beta][2\gamma], x) + c_2 x^{1-\gamma} F([2\alpha-2\gamma+2][2\beta-2\gamma+2][4-2\gamma], x).$$

It is this form which will be used in subsequent work, for the sake of avoiding fractional indices of q.

II.

Generalization of Appell's Theorem.

Consider a q-differential equation

$$\{q^{4\vartheta} + \lambda(x)\, q^{2\vartheta} + \mu(x)\}\, y = 0 \ldots\ldots\ldots\ldots (11),$$

in which $\lambda(x)$, $\mu(x)$ denote known functions. It is obvious that if $\lambda(x)$, $\mu(x)$ are functions of the form

$$(ax+b)/(cx+d) = \lambda(x),$$
$$(ex+f)/(cx+d) = \mu(x),$$

the equation (11) can be transformed into

$$\{x[\vartheta+\alpha][\vartheta+\beta]-[\vartheta][\vartheta+\gamma-1]\}\, y = 0 \ldots\ldots (12).$$

If now $\Phi(x)$ be any solution whatsoever, whether particular or general, of the above equation (11) we have, on substituting $\Phi(x)$ for y in the equation, and replacing

$$q^{2\vartheta}\Phi(x) \text{ by } \Phi(qx),$$
$$q^{2\vartheta}\Phi(x) \text{ by } \Phi(q^2 x),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

* *American Journal of Mathematics*, October, 1910.

an equation

$$\Phi(q^4x) + \lambda(x)\,\Phi(q^2x) + \mu(x)\,\Phi(x) = 0 \ldots\ldots (13).$$

We may consider this as a functional equation or as a special type of "difference" equation, namely, of multiplicative periodicity. To obtain a solution of such an equation is equivalent to solving the differential equation (11). Since x is variable, we may also write

$$\left.\begin{aligned} &\Phi(q^5x) + \lambda(qx)\ \Phi(q^3x) + \mu(qx)\ \Phi(qx) = 0 \\ &\Phi(q^6x) + \lambda(q^2x)\,\Phi(q^4x) + \mu(q^2x)\,\Phi(q^2x) = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots (13A).$$

Now it is quite easy to combine (13) and (14) into one equation (see appendix A), namely,

$$\begin{aligned} &\Phi(q^5x)\,\Phi(q^4x) + \lambda(qx)\,\Phi(q^4x)\,\Phi(q^3x) \\ &= \lambda(x)\,\mu(qx)\,\Phi(q^2x)\,\Phi(qx) \\ &\qquad + \mu(x)\,\mu(qx)\,\Phi(qx)\,\Phi(x) \ldots\ldots (14). \end{aligned}$$

We note that the indices of q under the sign Φ are pairs of consecutive integers, namely,

$$(5, 4),\ (4, 3),\ (2, 1),\ (1, 0) \ldots\ldots\ldots\ldots (15),$$

and it is this property which makes it possible for us to convert this functional equation into a differential equation satisfied by

$$\Phi(x)\,\Phi(qx).$$

For, since

$$\left.\begin{aligned} &q^5\Phi(x)\,\Phi(qx) = \Phi(qx)\,\Phi(q^3x) \\ &q^{25}\Phi(x)\,\Phi(qx) = \Phi(q^2x)\,\Phi(q^3x) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots (16),$$

we see that (14) is equivalent to

$$\{q^{45} + \lambda(qx)\,q^{33} - \lambda(x)\,\mu(qx)\,q^9 - \mu(x)\,\mu(qx)\}\,z = 0 \ldots (17).$$

with a solution $\qquad z = \Phi(x)\,\Phi(qx).$

This equation (17) is a generalization of Appell's equation, and, in the case ($\lim q \to 1$), may be transformed into that equation. Moreover, it is obvious that if

$$\lambda(qx), \quad \lambda(x)\,\mu(qx), \quad \mu(x)\,\mu(qx)$$

are each of the type

$$\frac{ax + b}{cx + d},$$

it may be possible (and, in fact, is so) to convert, by factorization, the equation (17) into the form

$$x(q^{\vartheta+\rho_1}-1)(q^{\vartheta+\rho_2}-1)(q^{2\vartheta+\rho_3}-1)y - (q^{2\vartheta+\sigma_1}-1)(q^{\vartheta+\sigma_2}-1)(q^{\vartheta}-1)y=0,$$

which is the same as

$$\{x[\vartheta+\rho_1][\vartheta+\rho_2][2\vartheta+\rho_3]+[\vartheta][2\vartheta+\sigma_1][\vartheta+\sigma_2]\}y=0,$$

namely, an equation satisfied by a q-hypergeometric series with five elements. In the next section of the paper we shall perform this transformation, and so obtain the q-generalization of Goursat's product of hypergeometric series.

III.

The special equation

$$\{x[2\vartheta+2\alpha][2\vartheta+2\beta]-[2\vartheta][2\vartheta+2\gamma-2]\}y=0 \ldots (18).$$

This equation comes under the general form

$$\{q^{4\vartheta}+\lambda(x)q^{2\vartheta}+\mu(x)\}y=0 \ldots\ldots\ldots (19);$$

as a special case

$$\left.\begin{aligned}\lambda(x)&=\frac{xq^{2\alpha}+xq^{2\beta}-q^{2\gamma-2}-1}{xq^{2\alpha+2\beta}-q^{2\gamma-2}}\\ \mu(x)&=\frac{x-1}{xq^{2\alpha+2\beta}-q^{2\gamma-2}}\end{aligned}\right\}\ldots\ldots\ldots\ldots (20).$$

This equation (18) has a principal solution

$$y=c_1\phi_1(x)+c_2\phi_2(x),$$

where ϕ_1, ϕ_2 denote q-hypergeometric series

$$\left.\begin{aligned}\phi_1&=1+\frac{[2\alpha][2\beta]}{[2][2\gamma]}x+\ldots\\ \phi_2&=1+\frac{[2\alpha-2\gamma+2][2\beta-2\gamma+2]}{[2][4-2\gamma]}x+\ldots\end{aligned}\right\}\ldots (21).$$

Similarly the equation

$$\{qx[2\vartheta+2\alpha][2\vartheta+2\beta]-[2\vartheta][2\vartheta+2\gamma-2]\}y=0,$$

which only differs from the preceding equation (18) in that qx replaces x, has a principal solution

$$a_1\phi_1(qx)+a_2\phi_2(qx)\ldots\ldots\ldots\ldots\ldots (22),$$

where $\phi_1(qx)$, $\phi_2(qx)$ are the q-hypergeometric series (21), with qx replacing x.

If, now, in the equation

$$\{q^{4\vartheta}+\lambda(qx)\,q^{3\vartheta}-\lambda(x)\,\mu(qx)\,q^{\vartheta}-\mu(x)\,\mu(qx)\}\,z=0\ldots(23),$$

we substitute the special values (20) of

$$\lambda(x),\quad \mu(x),\quad \mu(qx),\quad \lambda(qx),$$

and put $\gamma=\alpha+\beta+\frac{1}{2}$, we have

$$\lambda(x)\,\mu(qx)=\frac{xq^{2\alpha}+xq^{2\beta}-q^{2\alpha+2\beta-1}-1}{q^{1\alpha+4\beta}(x-q^{-2})},$$

$$\mu(x)\,\mu(qx)=\frac{x-1}{q^{4\alpha+4\beta}(x-q^{-2})},$$

and the equation (23) becomes

$$x\{q^{4\vartheta+4\alpha+4\beta}+q^{3\vartheta+4\alpha+2\beta}+q^{3\vartheta+2\alpha+1\beta}-q^{\vartheta+2\alpha}-q^{\vartheta+2\beta}-1\}\,z$$
$$-\{q^{4\vartheta+4\alpha+4\beta-2}+q^{3\vartheta+4\alpha+4\beta-2}+q^{3\vartheta+2\alpha+2\beta-1}$$
$$-q^{\vartheta+2\alpha+2\beta-1}-q^{\vartheta}-1\}\,z=0\ldots\ldots(24),$$

which, by factorization, is at once expressed as

$$x[\vartheta+2\alpha][\vartheta+2\beta][2\vartheta+2\alpha+2\beta]\,z$$
$$-[\vartheta][\vartheta+2\gamma-2][2\vartheta+2\gamma-2]\,z=0\ldots\ldots(25)$$

($\gamma=\alpha+\beta+\frac{1}{2}$). This is the q-analogue of Goursat's equation

$$x(\vartheta+2\alpha)(\vartheta+2\beta)(2\vartheta+2\alpha+2\beta)\,z$$
$$-(\vartheta)(\vartheta+2\gamma-2)(2\vartheta+2\gamma-2)\,z=0.$$

To solve equation (25) we substitute for z the series

$$\Sigma a_r x^{\theta+r},$$

and obtain an indicial equation

$$[\theta][2\theta+2\gamma-2][\theta+2\gamma-2]=0\ldots\ldots\ldots(26),$$

also a relation between successive coefficients

$$a_r=a_{r-1}\frac{[\theta+2\alpha+r-1][\theta+2\beta+r-1][2\theta+2\alpha+2\beta+2r-2]}{[\theta+r][2\theta+2\gamma+2r-2][\theta+2\gamma+r-2]},$$

The principal roots of the indicial equations are

$$\theta=0,\quad \theta=1-\gamma,\quad \theta=2-2\gamma\ldots\ldots\ldots\ldots(27),$$

so that we have as a principal solution

$$y = A\left\{1 + \frac{[2\alpha][2\beta]}{[1][2\gamma]}x + \ldots\right\}$$
$$+ Bx^{1-\gamma}\left\{1 + \frac{[2\alpha+1-\gamma][2\beta+1-\gamma][1]}{[2][2-\gamma][\gamma]}x + \ldots\right\}$$
$$+ Cx^{2-2\gamma}\left\{1 + \frac{[2\alpha+2-2\gamma][2\beta+2-2\gamma]}{[1][4-2\gamma]}x + \ldots\right\} \ldots (28),$$

in which $\gamma = \alpha + \beta + \frac{1}{2}$. Each of these three series is a q-hypergeometric series of five elements. It follows from the general result (17), of which the above is a special case, that

$$\left\{1 - \frac{[2\alpha][2\beta]}{[2][2\gamma]}x + \ldots\right\} \times \left\{1 + \frac{[2\alpha][2\beta]}{[2][2\gamma]}qx + \ldots\right\}$$
$$= \left\{1 + \frac{[2\alpha][2\beta]}{[1][2\gamma]}x + \frac{[2\alpha][2\alpha+1][2\beta][2\beta+1][2\gamma+1]}{[2\gamma][2\gamma][2\gamma+2][1][2]}x^2 + \ldots\right\}$$
$$\ldots\ldots (29).$$

This equation was communicated to me some months ago by Mr. G. N. Watson. He surmised its truth, having verified it as far as x^3. It is further obvious that if $\phi_1(x)$ denote

$$1 + \frac{[2\alpha][2\beta]}{[2][2\gamma]}x + \frac{[2\alpha][2\alpha+2][2\beta][2\beta+2]}{[2][4][2\gamma][2\gamma+2]}x^2 + \ldots,$$

and $\phi_2(x)$ denote

$$= 1 + \frac{[2\alpha-2\gamma+2][2\beta-2\gamma+2]}{[2][4-2\gamma]}x + \ldots,$$

then a relation

$$A\phi_1(x)\,\phi_2(qx) + B\phi_1(qx)\,\phi_2(x)$$
$$= 1 + \frac{[2\alpha+1-\gamma][2\beta+1-\gamma][1]}{[2][2-\gamma][\gamma]}x + \ldots \ldots\ldots (30)$$

must also exist, in which A and B are constants, such that $A + B = 1$. The equation then reduces to Goursat's result

$$F(\alpha, \beta, \gamma, x) \times F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x)$$
$$= F(2\alpha - \gamma + 1, 2\beta - \gamma + 1, \tfrac{1}{2}, 2 - \gamma, \gamma, x),$$

where $\gamma = \alpha + \beta + \frac{1}{2}$.

I do not propose to pursue the results any further. It is obvious that by this method all the results for q-series corresponding to those obtained by Goursat, Orr, and Watson for the ordinary hypergeometric series may be obtained.

Appendix A.

The equations (13), (13*A*) may be written

$$\Phi(q^4x) = -\lambda(x)\,\Phi(q^2x) - \mu(x)\,\Phi(x),$$

$$\Phi(q^5x) + \lambda(qx)\,\Phi(q^3x) = -\mu(qx)\,\Phi(qx).$$

The product of these gives at once

$$\Phi(q^5x)\,\Phi(q^4x) + \lambda(qx)\,\Phi(q^4x)\,\Phi(q^3x) - \lambda(x)\,\mu(qx)\,\Phi(q^2x)\,\Phi(qx) - \mu(qx)\,\mu(x)\,\Phi(qx)\,\Phi(x) = 0,$$

in which the indices of q, under the functional sign Φ, are pairs of successive integers; hence the possibility of transforming the functional equation into a differential equation

$$\{q^{4z} + \lambda(qx)\,q^{3z} - \lambda(x)\,\mu(qx)\,q^{z} - \mu(x)\,\mu(qx)\}\,z = 0,$$

satisfied by $\Phi(x)\,\Phi(qx)$. We see that, in order that the transformation may be possible, we must be able to combine two of the equations (13), (13*A*) into an equation where the indices of q under sign Φ are pairs of consecutive integers: This is possible, and so we have a q-generalization of Appell's equation. The question then arises: is it possible from three of the equations (13) to form a new equation, in which the indices of q, under the sign Φ, shall form sets of three consecutive integers; for example, to form such an expression as

$$\Phi(q^6x)\,\Phi(q^5x)\,\Phi(q^4x) + a(x)\,\Phi(q^5x)\,\Phi(q^4x)\,\Phi(q^3x) + \text{etc.} = 0.$$

This, in general, is not possible; I think the only possible case is that of pairs of consecutive integers; I surmise that this result involves the impossibility of extending Goursat's product of two hypergeometric series to the case of three or more hypergeometric series.

ON A THEOREM IN THE HARNACK INTEGRATION OF SERIES.

By *W. H. Young, Sc.D., F.R.S.*

§ 1. WHILE the theory of the integration of sequences of unbounded functions integrable in the Lebesgue sense is rapidly reaching what appears to be its final form, something still remains to be done when the integration is possible only in Harnack's sense or more generally in what may be called the Harnack-Lebesgue sense. Harnack integrals have, however, as is well known, almost equal importance with Lebesgue integrals, and naturally arise in the process of analysis.

Some of the results proved for Lebesgue integration are easily seen to hold equally for Harnack integration. Thus, *if the sequences of which the general terms are g_n and h_n are integrable in the Harnack manner or Harnack-Lebesgue manner, and f_n lies between g_n and h_n for all values of n and x, then the f_n-sequence is integrable in the same manner.* This follows from the fact that the sequence $(h_n - g_n)$ is integrable in the Lebesgue manner, as all the functions of this sequence have the same sign for all values of n and x. Hence the sequence $(h_n - f_n)$ is integrable in the Lebesgue sense,* and therefore the sequence f_n is integrable in the Harnack sense. As an immediate corollary, we have of course the theorem that monotone sequences are integrable in the Harnack sense if the first function and also the limiting function (and therefore the remaining functions of the sequence) possess Harnack integrals.

The object of the present note is to prove the following theorem in the Harnack integration of series:—

Let $f_1, f_2, \ldots$ be a bounded sequence of monotone positive functions of x, having $f(x)$ as limiting function, and let $g(x)$ be any function having a Harnack integral in the interval (a, b). Then the sequence $f_n(x)g(x)$ is integrable term by term in the Harnack manner.

This theorem is then applied, as an example of its usefulness, to prove the following theorem in double limits:—

If $f(x, n)$ is a positive monotone decreasing function of x, and $g(x)$ is a function having a Harnack integral in the infinite

* W. H. Young, "On Semi-Integrals and Oscillating Successions of Functions," 1910, presented to the *London Math. Soc.*

interval (a, ∞), *then* $\int_a^p f(x,n)\,g(x)\,dx$ *has, as* n *and* p *approach infinity in any manner whatever, an unique double limit, provided only that the functions* $f(x, n)$ *form a bounded sequence as* n *increases.**

§ 2. Theorem 1. *Let* $f_1, f_2, \ldots$ *be a sequence of monotone decreasing positive functions of* x, *having* $f(x)$ *as limiting function in the interval* (a, b). *Let this sequence be bounded, or, which is the same thing, let the upper bound of* $f_n(a)$ *be a finite quantity* M. *Let* $g(x)$ *be any (finite) function having a Harnack interval in* (a, b), *then the sequence* $f_n(x)\,g(x)$ *is integrable term by term in the Harnack manner.*

First let there be no infinite discontinuities of $g(x)$ internal to the interval (a, b). The theorem is then true in any interval (p, q) internal to the completely open interval (a, b), since the sequence $f_n(x)\,g(x)$ is bounded in (p, q). We can then show, as follows, that the theorem is true in (a, b). For, since $g(x)$ has a Harnack integral over (a, b), we can, having pre-assigned a positive quantity e, determine p and q so that, if d_p denotes any interval less than (a, p) with a as left-hand end-point, and p as right-hand end-point, and d_q any interval less than (q, b) with q as left-hand and b as right-hand end-point,

$$\left| \int_p^q g(x)\,dx - \int_a^b g(x)\,dx \right| < e,$$

whence
$$-e < \int_{d_p} g(x)\,dx + \int_{d_q} g(x)\,dx < e.$$

Since this is true as q moves up to b as limit,

$$-e < \int_{d_p} g(x)\,dx < e.$$

Hence, by the Second Theorem of the Mean,

$$-e f_n(a) < \int_{d_p} f_n(x)\,g(x)\,dx < e f_n(a),$$

* A less general statement of this theorem, with the explicit limitation that $f(x, n)$ should tend to its limit uniformly and the tacit assumption that $g(x)$ is bounded, is given on p. 443 of Bromwich's *Theory of Infinite Series.*

whence, for all values of n,

$$-Me < \int_{d_p} f_n(x)\, g(x)\, dx < Me.$$

Similarly $$-Me < \int_{d_q} f_n(x)\, g(x)\, dx < Me,$$

so that

$$-2Me + \int_p^q f_n(x)\, g(x)\, dx < \int_a^b f_n(x)\, g(x)\, dx < 2Me + \int_p^q f_n(x)\, g(x)\, dx.$$

Now, as we saw, the theorem holds in the interval (p, q). Thus we get, proceeding to the limit with n,

$$-2Me + \int_p^q f(x)\, g(x)\, dx \leq \operatorname*{Llt}_{n=\infty} \int_a^b f_n(x)\, g(x)\, dx \leq 2Me + \int_p^q f(x)\, g(x)\, dx.$$

But, since $g(x)$ has a Harnack integral from a to b, and $f(x)$ being the limit of a sequence of positive monotone descending functions is a positive monotone descending function, $f(x)g(x)$ is by a known theorem* integrable in the Harnack manner. Hence, letting p and q move up to their respective limiting points a and b,

$$-2Me + \int_a^b f(x)\, g(x)\, dx \leq \operatorname{Lt} \int_a^b f_n(x)\, g(x)\, dx \leq 2Me + \int_a^b f(x)\, g(x)\, dx.$$

Since this is true for all positive values of e, however small, we see that all the limits in the middle coincide and the value of the unique limit so obtained is the integral of $f(x)\, g(x)$ from a to b. That is, the sequence $f_n(x)\, g(x)$ is integrable in the Harnack manner from a to b.

Next let there be infinite discontinuities of $g(x)$ internal to (a, b), these then form a closed set of zero content, since $g(x)$ has a Harnack integral.

Let (a_1, b_1), (a_2, b_2), ..., (a_r, b_r), ... be the black intervals

* E. W. Hobson, "The Second Mean-Value Theorem of the Integral Calculus," *Proc. L.M.S.*, vol. vii. (1908), series 2, p. 21.

of this closed set. Then, since there are no infinite discontinuities of $g(x)$ internal to (a_r, b_r), the theorem holds, by what precedes, in each interval (a_r, b_r). Hence, fixing any integer m,

$$\sum_{r=1}^{r=m} \operatorname{Lt}_{n=\infty} \int_{a_r}^{b_r} f_n(x)\, g(x)\, dx = \sum_{r=1}^{r=m} \int_{a_r}^{b_r} f(x)\, g(x)\, dx,$$

or, since there is only a finite number of terms in the summation,

$$\operatorname{Lt}_{n=\infty} \sum_{r=1}^{r=m} \int_{a_r}^{b_r} f_n(x)\, g(x)\, dx = \sum_{r=1}^{r=m} \int_{a_r}^{b_r} f(x)\, g(x)\, dx \ldots (1).$$

Now, if w_r denote the oscillation of the integral $\int_{a_r}^{x} g(x)\,dx$, in the closed interval (a_r, b_r), we know, by a theorem of E. H. Moore's,† that the w's form a convergent series, so that the moduli of the integrals $\int_{a_r}^{b_r} g(x)\,dx$ also form such a series. Let m be chosen so large that the remainders of each of these convergent series after the m^{th} terms are numerically less than e.

By the second Theorem of the Mean, we have

$$f_n(a_r)\, h_r \leq \int_{a_r}^{b_r} f_n(x)\, g(x)\, dx \leq f_n(a_r)\, H_r,$$

where $$H_r - h_r = w_r.$$

Hence the integral in the middle of the preceding inequality differs from $f_n(a_r) \int_{a_r}^{b_r} g(x)\,dx$ by less than $f_n(a_r)\, w_r$, that is by less than Mw_r. Using G_r for the absolute value of the integral of $g(x)$ over (a_r, b_r), it follows that

$$\left| \int_{a_r}^{b_r} f_n(x)\, g(x)\, dx \right| \leq M[G_r + w_r].$$

Therefore, summing for all values of r greater than m,

$$\sum_{r=m+1}^{\infty} \left| \int_{a_r}^{b_r} f_n(x)\, g(x)\, dx \right| \leq 2Me.$$

Hence also $$-2Me \leq \sum_{r=m+1}^{\infty} \int_{a_r}^{b_r} f_n(x)\, g(x)\, dx \leq 2Me.$$

† E. H. Moore, *Trans. Amer. Math. Soc.*, vol. ii., p. 296 and 459, quoted without proof in Hobson's Treatise, p. 384.

Proceeding to the limit with n, the limits of the summation in the middle are numerically not greater than $2Me$. Using (1), therefore, we get

$$-2Me + \sum_{r=1}^{r=m} \int_{a_r}^{b_r} f(x)\, g(x)\, dx \leq \operatorname{Lt}_{n=\infty} \sum_{r=1}^{\infty} \int_{a_r}^{b_r} f_n(x)\, g(x)\, dx$$

$$\leq 2Me + \sum_{r=1}^{r=m} \int_{a_r}^{b_r} f(x)\, g(x)\, dx.$$

But, since $f_n(x)\, g(x)$ has a Harnack integral, the summation in the middle is equal to $\int_a^b f_n(x)\, g(x)\, dx$ by Moore's theorem; also, letting e decrease indefinitely, m increases indefinitely and the extremes in the preceding inequality have the same unique limit, viz., $\int_a^b f(x)\, g(x)\, dx$, by Moore's theorem, since the integrand has a Harnack integral. Thus the limits in the middle of the preceding inequality all coincide and we get

$$\operatorname{Lt}_{n=\infty} \int_a^b f_n(x)\, g(x)\, dx = \int_a^b f(x)\, g(x)\, dx,$$

which proves the theorem.

§ 3. Theorem 2. *If $f(x, n)$ is a positive monotone decreasing function of x and $g(x)$ is a function having a Harnack integral in the interval (a, ∞), then*

$$\int_a^p f(x, n)\, g(x)\, dx$$

has, as n and p approach infinity in any manner whatever, an unique double limit, provided only that the functions $f(x, n)$ form a bounded sequence as n increases, in other words, provided that $f(x, n)$ has an unique limit $f(x)$ and $f(a, n)$ has a finite upper bound M.

For,* then, by the Second Theorem of the Mean for Harnack integrals, we have

$$\left| \int_z^p f(x, n)\, g(x)\, dx \right| \gtrless f(z, n)\, H \gtrless MH,$$

* It will be seen that this reasoning is essentially the same as that given by Bromwich, *loc. cit.* The chief alteration is that Theorem 1 of the present paper is used instead of the special case of it then known.

where H is the upper bound of $\int_z^{z'} g(x)\,dx$, as z' ranges from z to infinity. Since the improper integral of $g(x)$ exists in the interval (a, ∞), we can, pre-assigning any positive e, find z so that H is less than e.

Again, since each function $f(x, n)$ is positive and monotone increasing and $\leq M$, the same is true of the limiting function $f(x)$, so that, by the same reasoning,

$$\int_z^p f(x)\,g(x)\,dx \leq ME.$$

Hence

$$\left|\int_a^p f(x,n)\,g(x)\,dx - \int_a^p f(x)\,g(x)\,dx\right| \leq 2Me + \left|\int_a^z \{f(x,n)-f(x)\}\,g(x)\,dx\right|.$$

Now, by Theorem 1 of the present paper, the integral on the right has zero as unique limit when n is indefinitely increased, z remaining constant, since $\{f(x, n)-f(x)\}$ converges to zero and $g(x)$ is, by definition of its improper integral, certainly integrable in the Harnack manner in the interval (a, z). Since this right-hand side is independent of p, we get the result that all the double limits of the left-hand side of the preceding inequality are less than $2Me$. Hence, e being at our disposal, the left-hand side has zero as unique limit.

But $\int_a^p f(x)\,g(x)\,dx$ has, as p increases indefinitely, the unique limit $\int_a^\infty f(x)\,g(x)\,dx$, so that $\int_a^p f(x,n)\,g(x)\,dx$ has also an unique double limit as n and p increase indefinitely in any manner whatever, and this unique double limit is $\int_a^\infty f(x)\,g(x)\,dx$, which proves the theorem.

§ 4. In conclusion we may remark that with slight modification the argument here used shows that the theorems here proved are true for Harnack-Lebesgue integration.

Geneva,
August 29th, 1910.

NOTE ON GORDAN'S PROOF OF HERMITE'S LAW OF RECIPROCITY.

By *H. T. H. Piaggio.*

A PROOF that depends upon the symbolical notation has been given by Gordan* of Hermite's well-known theorem that the number of rational integral seminvariants of weight w, degree i, and extent not exceeding p, is equal to the number of rational integral seminvariants of weight w, degree p, and extent not exceeding i.

A shorter form of this proof, given by Elliott† to illustrate the advantage of his modified symbolical notation, merely consists of drawing attention to the formal analogy between

$$\Sigma\,(\alpha_1-\alpha_2)^{\lambda}(\alpha_1-\alpha_3)^{\mu}\ldots(\alpha_r-\alpha_s)^{\sigma}\ldots$$

(where $\alpha_1\alpha_2\ldots\alpha_p$ are the roots of a p^{ic}, $\lambda+\mu+\ldots+\sigma+\ldots=w$, and i is the highest power to which any one α occurs) and

$$(\theta_1-\theta_2)^{\lambda}(\theta_1-\theta_3)^{\mu}\ldots(\theta_r-\theta_s)^{\sigma}\ldots,$$

acting on $\qquad a_0^{(1)}a_0^{(2)}\ldots a_0^{(p)},$

where the θ's are operators defined by

$$\theta_r \equiv a_1^{(r)}\frac{\partial}{\partial a_0^{(r)}}+a_2^{(r)}\frac{\partial}{\partial a_1^{(r)}}+\ldots+a_i^{(r)}\frac{\partial}{\partial a_{i-1}^{(r)}}.$$

After operation we put $a_s^{(1)}=a_s^{(2)}=\ldots=a_s^{(p)}=a_s$ for all values of s.

Now many seminvariant types vanish when two or more of the sets of symbols refer to the same quantic; for example, (ab) or $(\theta_1-\theta_2)\,a_0^{(1)}a_0^{(2)}$. The question has been raised whether, when a seminvariant produced by the θ's vanishes in consequence of this equivalence of symbols, the corresponding α seminvariant is necessarily zero (and conversely). If not, Gordan's proof seems to be invalidated.

The following reasoning shows that if either is zero so is the other, and so the proof is valid.

(A) If the function of the α's, which we will call $\Sigma\Delta\,(\alpha)$, vanishes, then the result of operating with $\Delta\,(\theta)$ on

$$A \equiv a_0^{(1)}a_0^{(2)}\ldots a_0^{(p)}$$

and making the p sets of symbols equivalent after operation also vanishes.

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* *Invariantentheorie*, band. ii., pp. 97–98.

† In a paper entitled "Perpetuants and Contra-perpetuants," *Proc. L.M.S.*, series ii., vol. iv.

This part is obvious, for if $\Sigma\Delta(\alpha)=0$ identically, $\Sigma\Delta(\theta)=0$ identically, hence

$$\Delta(\theta).A=\frac{1}{p!}\Sigma\Delta(\theta).A=\frac{1}{p!}.\text{zero}.A=0.$$

(*B*) If $\Delta(\theta).A$ vanishes when the sets of symbols are made equivalent, then $\Sigma\Delta(\alpha)$ vanishes identically.

For equivalent sets of symbols, if $\Delta(\theta).A=0$, so also

$$\Sigma\Delta(\theta).A=p!\,\Delta(\theta).A=0.$$

Suppose a term $\kappa\theta_1^{\lambda}\theta_2^{\mu}\theta_3^{\nu}\ldots$ occurs in $\Sigma\Delta(\theta)$, κ being a numerical coefficient. This term operating on A will give $a_{\lambda}^{(1)}a_{\mu}^{(2)}a_{\nu}^{(3)}\ldots$, *i.e.*, $a_{\lambda}a_{\mu}a_{\nu}\ldots$, on making the sets equivalent. The only other terms of $\Sigma\Delta(\theta)$ which give $a_{\lambda}a_{\mu}a_{\nu}\ldots$ are those derived from $\theta_1^{\lambda}\theta_2^{\mu}\theta_3^{\nu}\ldots$ by interchange of suffixes, and from the symmetry of the Σ each of these occurs with the same coefficient κ. Hence coefficient of $a_{\lambda}a_{\mu}a_{\nu}\ldots=p!\,\kappa$. If this coefficient vanishes, κ must vanish; similarly for all the other terms of $\Sigma\Delta(\theta)$. We have now proved that if $\Delta(\theta)$ and hence $\Sigma\Delta(\theta)$, operating on A, produce zero when the sets of symbols are made equivalent, then $\Sigma\Delta(\theta)$ must be itself identically zero; or, writing α_r for θ_r, $\Sigma\Delta(\alpha)$ is identically zero. Q.E.D.

We can deal similarly with the case in which p_1 of the θ's refer to one quantic of order i_1, p_2 to one of order i_2, and so on. In this case the Σ will denote the sum of $\dfrac{p!}{p_1!\,p_2!\ldots}$ terms.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy*, Trinity College, Cambridge.

XXX.

A theorem concerning summable integrals.

§ 1. The following theorem, concerning series summable by Cesàro's method of mean values, was proved independently by H. Bohr and myself.*

If (i) Σa_n *is summable* (Cr), (ii) $f_n\to 0$, (iii) *the series*

$$\Sigma u^r\,|\,\Delta^{r+1}f_n\,|$$

is convergent, then the series $\Sigma a_n f_n$ *is summable* (Cr), *and its sum is equal to that of the absolutely convergent series*

$$\Sigma S_n^{\,r}\Delta^{r+1}f_n.$$

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* See *Proc. Lond. Math. Soc.*, vol. vi. (1908), p. 257, and vol. viii. (1910), p. 277; and Bohr, *Comptes Rendus*, January 11, 1909, and *Bidrag til de Dirichlet'ske Raekkers Theorie*, Copenhagen 1910.

Here S_n^r is defined by the eqnations

$$S_n = a_0 + a_1 + \ldots + a_n, \quad S_n^1 = s_0 + s_1 + \ldots + s_n,$$
$$S_n^k = S_0^{k-1} + S_1^{k-1} + \ldots + S_n^{k-1},$$

so that
$$r!\, S_n^r / n^r \to s,$$

where s is Cesàro's sum of the series Σa_n.

§ 2. This theorem suggests the truth of an analogous theorem for integrals.

Suppose $f(x)$ integrable and absolutely integrable throughout any finite interval, and let

$$f_1(x) = \int_a^x f(t)\, dt, \quad f_2(x) = \int_a^x f_1(t)\, dt, \ \ldots,$$

so that we may write symbolically

$$f_k(x) = \left(\int_a^x dt\right)^k f(t),$$

then we shall say that the integral

$$\int_a^\infty f(x)\, dx \tag{1}$$

is *summable* (r, x)* to sum s if

$$r!\, f_{r+1}(x) / x^r \to s \tag{2}$$

as $x \to \infty$. In virtue of a well-known formula† this definition may also be presented in the form

$$\int_a^x \left(1 - \frac{t}{x}\right)^r f(t)\, dt \to s. \tag{2'}$$

The theorem then runs as follows:—*Suppose that* (i) *the integral* (1) *is summable* (r, x), (ii) $\phi(x)$ *has a derivative* $\phi^{(r+1)}(x)$ *integrable and absolutely integrable in any finite interval*, (iii) *the integral*

$$\int_a^\infty x^r \left|\, \phi^{(r+1)}(x) \,\right| dx$$

is convergent, (iv) $\phi(x) \to 0$ *as* $x \to \infty$; *then will the integral*

$$\int_a^\infty f(x)\, \phi(x)\, dx \tag{3}$$

be summable (r, x), *and its sum will be equal to the value of the absolutely convergent integral*

$$(-1)^{r+1} \int_a^\infty f_{r+1}(x)\, \phi^{(r+1)}(x)\, dx. \tag{4}$$

* I hope in a later note to develop more systematically some properties of integrals summable by this and by more general methods: the reason for this notation will then appear.

† Jordan, *Cours d'Analyse*, t. 3, p. 59.

It is convenient to suppose $a > 0$, and it is easy to see that no generality is lost by this assumption.

In order to prove this theorem we require certain lemmas.

§ 3. Lemma 1. *If the integral* (1) *is summable* (r, x) *to sum* s, *it is summable* $(r+1, x)$ *to the same sum. If it is convergent and has the value* s, *it is summable* (r, x) *to sum* s *for any value of* r.

In fact, from (2), it follows, by the use of familiar theorems, that

$$(r+1)!\, f_{r+2}(x)/x^{r+1} \to s.$$

Lemma 2. *The formula*

$$\left(\int_a^x dt\right)^r (f\phi) = f_r\phi - \binom{r}{1}\left(\int_a^x dt\right)(f_r\phi') + \binom{r}{2}\left(\int_a^x dt\right)^2 (f_r\phi'') - \ldots + (-1)^r \left(\int_a^x dt\right)^r (f_r\phi^{(r)})$$

is true for all positive integral values of r.

To prove this we observe that

$$\left(\int_a^x dt\right)^{k+1} (f_r\phi^{(k)}) = \left(\int_a^x dt\right)^k \left(\int_a^x f_r\phi^{(k)}\, dt\right)$$

$$= \left(\int_a^x dt\right)^k \left(f_{r+1}\phi^{(k)} - \int_a^x f_{r+1}\phi^{(k+1)}\, dt\right)$$

$$= \left(\int_a^x dt\right)^k (f_{r+1}\phi^{(k)}) - \left(\int_a^x dt\right)^{k+1} (f_{r+1}\phi^{(k+1)}).$$

If now we assume the truth of the lemma for $r = s$, integrate once more, and apply the formula just proved to each term, we find that the lemma is true for $r = s+1$. As it is plainly true for $r = 1$, it is therefore true in general.

Lemma 3. *The conditions* (iii) *and* (iv) *of the theorem imply*

$$\phi' \to 0,\ \phi'' \to 0,\ \ldots,\ \phi^{(r)} \to 0,$$

and the convergence of all the integrals

$$\int_a^\infty x^{r-\rho}\, |\phi^{(r-\rho+1)}(x)|\, dx$$

$(\rho = 1, 2, \ldots, r)$. This was shown in Note xxix.

Lemma 4. *If* $\chi(x)$ *is limited and integrable in any finite interval, and tends to zero as* $x \to \infty$, *and* α *and* β *are real and greater than* -1, *then*

$$x^{-\alpha-\beta-1} \int_a^x t^\alpha (x-t)^\beta \chi(t)\, dt \to 0.$$

If we put $t = xu$, we obtain

$$\int_{a/x}^{1} u^a (1-u)^\beta \chi(xu)\, du,$$

which is numerically less than

$$\int_0^1 u^a (1-u)^\beta |\chi(xu)|\, du.$$

Choose τ so that $|\chi(t)| < \epsilon$ for $t \geqq \tau$. When τ is fixed we have

$$|\chi(xu)| < K \quad (0 \leqq u \leqq \tau/x).$$

Hence

$$\int_0^1 u^a (1-u)^\beta |\chi(xu)|\, du < K \int_0^{\tau/x} u^a du + \epsilon \int_0^1 u^a (1-u)^\beta du,$$

which is less than a constant multiple of ϵ when x is large enough.

§ 4. We can now proceed to the proof of the theorem. We have

$$\frac{r!}{x^r}\left(\int_a^x dt\right)^{r+1}(f\phi) = \frac{r!}{x^r}\sum_{\nu=0}^{r+1}(-1)^\nu \binom{r+1}{\nu}\left(\int_a^x dt\right)^\nu (f_{r+1}\phi^{(\nu)})$$

$$= \frac{r!}{x^r}\sum_{\nu=0}^{r+1}(-1)^\nu \binom{r+1}{\nu}\eta_\nu,$$

say, by Lemma 2. Here the operation $\left(\int_a^x dt\right)^0$, operating on a function of t, is to be interpreted as a substitution of x for t.

Now the integral (4) is absolutely convergent, since $|f_{r+1}(x)| < Kx^r$. It follows from Lemma 1 that

$$\frac{r!}{x^r}\eta_{r+1} \to \int_a^\infty f_{r+1}(x)\,\phi^{(r+1)}(x)\, dx.$$

In order to complete the proof of the theorem, we have therefore only to show that

$$\eta_\nu / x^r \to 0, \quad (\nu = 0, 1, \ldots, r),$$

In the first place

$$|\eta_0|/x^r = |f_{r+1}(x)\,\phi(x)/x^r| < K|\phi(x)| \to 0.$$

When $\nu > 0$,

$$\eta_\nu = \left(\int_a^x dt\right)^\nu (f_{r+1}\phi^{(\nu)}) = \frac{1}{(\nu-1)!}\int_a^x (x-t)^{\nu-1} f_{r+1}\phi^{(\nu)}\, dt,$$

which is numerically less than a constant multiple of

$$\zeta_\nu = \int_a^x (x-t)^{\nu-1} t^r |\phi^{(\nu)}|\, dt.$$

Let $$\Phi_\nu(t) = \int_t^\infty u^{\nu-1} |\phi^{(\nu)}(u)|\, du;$$

the convergence of this integral follows from Lemma 3. Then if $\nu = 1$, we have

$$\zeta_1 = -\int_a^x t^r \Phi_1'(t)\, dt = a^r \Phi_1(a) - x^r \Phi_1(x) + r\int_a^x t^{r-1}\Phi_1(t)\, dt,$$

and so $\zeta_1/x^r \to 0$. If $\nu > 1$, we have

$$\zeta_\nu = -\int_a^x (x-t)^{\nu-1} t^{r-\nu+1} \Phi_\nu'(t)\, dt$$

$$= (x-a)^{\nu-1} a^{r-\nu+1} \Phi_\nu(a) - (\nu-1)\int_a^x (x-t)^{\nu-2} t^{r-\nu+1} \Phi_\nu(t)\, dt$$

$$+ (r-\nu+1)\int_a^x (x-t)^{\nu-1} t^{r-\nu} \Phi_\nu(t)\, dt = \zeta_{\nu,1} + \zeta_{\nu,2} + \zeta_{\nu,3},$$

say. It is evident that $\zeta_{\nu,1}/x^r \to 0$, and that $\zeta_{\nu,2}/x^r$ and $\zeta_{\nu,3}/x^r$ also tend to zero follows from Lemma 4, by putting first

$$\alpha = \nu - 2, \quad \beta = r - \nu + 1, \quad \alpha + \beta + 1 = r,$$

and then

$$\alpha = \nu - 1, \quad \beta = r - \nu, \quad \alpha + \beta + 1 = r.$$

The theorem is thus established.

§ 5. Suppose, in particular,

$$\phi(x) = x^{-s},$$

where s is complex and has its real part positive. Then

$$\phi^{(r+1)}(x) = (-1)^{r+1} s(s+1)\ldots(s+r)\, x^{-s-r-1},$$

and the conditions of the theorem are plainly satisfied. It follows that *if the integral*

$$\int_a^\infty x^{-s} g(x)\, dx \quad (a > 0)$$

is summable (r, x) *for any particular value of* s, *then it is summable* (r, x) *for all values of* s *whose real part is greater. From this we deduce the existence of lines of summability*

$$R(s) = s_0, \quad R(s) = s_1, \quad R(s) = s_2, \ \ldots$$

such that $$s_r - 1 \leqq s_{r+1} \leqq s_r;$$

the integral (5) is summable (r, x) to the right of the line $R(s) = s_r$. All this corresponds precisely to what Bohr has proved concerning Dirichlet's series.

SIMULTANEOUS COMPLEX VARIABLES AND THEIR GEOMETRICAL REPRESENTATION.

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1. In the development of several branches of the theory of functions of a single complex variable, two distinct notions are of persistent recurrence. Thus if a variable w be related to another variable z by a relation $f(w, z) = 0$, of any form, we usually consider that w is defined as a function of z by the relation. But, frequently, there is ample reason for regarding z as a function of w; and important results, specially in connection with periodic functions, are obtained by using this notion of inversion. Again, it proves at once convenient and useful to associate the geometrical representation of the variables with the corresponding analysis: and through the fact that the geometry of the representation of z and of w in separate planes is simple and is intuitively suggestive, this notion of geometrical interpretation assumes wide and fruitful importance.

When we pass to the consideration of functions of complex variables more than one in number, difficulties in using these two notions of inversion and of geometrical representation are met at the very outset. They will be sufficiently indicated in their elementary bearings by dealing with the case when the number of independent variables is two.

2. We denote the two independent complex variables by z and z', where, when it is desirable to put in evidence their real and their imaginary parts, we write

$$z = x + iy, \qquad z' = x' + iy'.$$

A function w of z and z' can be regarded as given by a relation

$$f(w, z, z') = 0,$$

any precise form of f not being important for the present purpose.

Now a subsidiary use of the notion of inversion can be applied to this equation. Just as in the Cartesian equation of a surface in ordinary space it is practically a matter of indifference which of the three coordinates is to be regarded as expressed by the equation in terms of the other two, so here in the equation $f(w, z, z') = 0$ we might regard any one of the three complex quantities to be thus defined as a function

of the other two. But these two modes of regarding one and the same functional equation of this type are not a regular rendering of the notion of inversion, which requires the expression of the quantities originally regarded as independent in terms of the quantities originally regarded as dependent. In the present case, it is not conceivable that z and z' shall be expressible in terms of w when only a single functional equation is given.

But the difficulty, in its initial form at least, disappears when a couple of distinct functions, say w and w', occur. This occurrence might arise through the existence of two functional relations

$$f(w, z, z') = 0, \qquad g(w', z, z') = 0,$$

or, more generally, of two functional relations

$$F(w, w', z, z') = 0, \quad G(w, w', z, z') = 0,$$

the two forms being equivalent to one another when all the functions are algebraical in w and w'. We assume that the equations do actually define distinct functions w and w'; that is to say, we assume that the Jacobian

$$J\left(\frac{F, G}{w, w'}\right)$$

does not vanish identically. Further, w and w' are to be independent functions of z and z'; and therefore the Jacobian

$$J\left(\frac{w, w'}{z, z'}\right)$$

must not vanish identically. But

$$J\left(\frac{w, w'}{z, z'}\right) J\left(\frac{z, z'}{w, w'}\right) = 1,$$

$$J\left(\frac{F, G}{w, w'}\right) J\left(\frac{w, w'}{z, z'}\right) = J\left(\frac{F, G}{z, z'}\right),$$

and therefore neither of the Jacobians

$$J\left(\frac{z, z'}{w, w'}\right), \quad J\left(\frac{F, G}{z, z'}\right)$$

can vanish. In other words, we can regard the two relations $F = 0$ and $G = 0$ in another way; they define z and z' as two distinct functions of the two distinct variables w and w'. Thus, by the introduction of as many distinct and independent

functions w as there are independent variables, we are in a position to use the notion of inversion. But it must be borne in mind that the inversion should be equally possible from any two relations which are the exact and complete equivalent of the two relations $F=0$ and $G=0$ in any form that may be given initially.

Finally, we could regard any two of the four variables as independent and the other two as dependent.

3. The association of an adequate geometrical representation with the variation of two complex variables is a more difficult matter, and it does not appear to admit of a unique solution as satisfactory as in the case of a single variable. For the doubly infinite possibilities of z, $=x+iy$, an adequate field is provided on a surface in ordinary space, and particularly on a plane. For the quadruply infinite possibilities of z, $=x+iy$, and of z', $=x'+iy'$, varying independently, the requirements cannot easily be provided by the facilities offered in the ordinary space of our experience. Several methods have been adopted or proposed: any measure of completeness is attained through some greater or smaller amount of elaboration; but with increase of elaboration there comes a decrease of simplicity (and therefore also a decrease of suggestive usefulness) in the geometry.

4. Of these methods, the most simple (in a formal analytical bearing) is the employment of multi-dimensional space; and applications to the theory of functions of two complex variables, by means of space of four dimensions, have been made by Poincaré,* Picard,† and others. The four real variables x, y, x', y' are associated with four axes of reference: sometimes they are taken as the ultimate variables of reference: sometimes they are made functions of other ultimate (and real) variables of reference, from one to four in number according to the dimensions of the continuum to be represented. Thus a single relation between x, y, x', y' provides a hypersurface in the multiple space: two such relations provide a surface: three of them provide a line: four of them provide a point (or a number of surfaces, lines, points, as the case may be). The intersection of a hypersurface and a surface is made up of a line or a number of lines: two surfaces intersect in points:

* *Acta Math.*, t. ii. (1883), pp. 97–113; *ib.*, t. ix. (1887), pp. 321–380; and in several later memoirs.

† References are given in Picard's *Théorie des fonctions algébriques de deux variables indépendantes*, t. i., ch. ii.

but, having regard to the source and the purpose of the geometrical representation, we consider only real lines and real points as they occur in such intersections.

At first sight, it would seem as if the representation were not sufficiently definite and restricted; for there is no preferential combination among the coordinate axes, while x and y are tied together into one complex variable, and x' and y' are tied together into the other. But this lack of restriction is supplied, in so far as functions of z and z' are concerned, by retaining the partial differential equations of the first order which are satisfied by the real and the imaginary parts of any function w. When we write

$$w = u + iv = f(z, z'),$$

we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial x'} = \frac{\partial v}{\partial y'}, \quad \frac{\partial u}{\partial y'} = -\frac{\partial v}{\partial x'},$$

so that u satisfies (as does v also) the equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x \partial y'} - \frac{\partial^2 u}{\partial y \partial x'} = 0,$$

$$\frac{\partial^2 u}{\partial x \partial x'} + \frac{\partial^2 u}{\partial y \partial y'} = 0, \qquad \frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2} = 0.$$

From a value of u, satisfying these equations, a value of v is deducible. In both of them, the proper combinations of x and y and of x' and y' occur.

For the present purpose it is unnecessary to specify in detail the results obtained by Poincaré in the extension of one of Cauchy's integral theorems, the formation of the residues of double integrals, in the topology of multiple space, and in the conformation of spaces. The comparative lack of familiarity with the ideas of four-dimensional geometry prevents this mode of representation from being useful *ab initio*; and, in particular, it does not provide (what other methods equally fail to provide) a representation of quadruple periodicity which serves the same kind of purpose as is served by the plane representation of double periodicity.

5. Another method of representation of the four elements of two complex variables is based upon the fact that four coordinates are needed for the complete specification of a straight line in ordinary space. Such a line would be determined uniquely by the two points at which it meets a couple

of parallel planes; and therefore, if one variable z were represented in one of the planes and the other variable z' were represented in the other plane, we might regard the line joining the points z and z' in the respective planes as representing the two variables z and z' conjointly. Should this representation be adopted, care must be exercised lest assumptions be made that are not justified by the facts; and in this case the facts are that two points on the line represent the two variables, and that two points uniquely determine a straight line.

If, however, the whole line (with the assistance of the two invariable parallel planes of reference) be taken to represent the two variables, a question at once arises as to the relations between a line z, z' and a line w, w', when the four variables are connected by two relations. Does the whole line z, z', under any transforming relations, become the whole line w, w'? The negative result, viz., that a straight line is not thus generally transformed into a straight line, can be established as follows:—

For simplicity, assume that the planes for z and z' are at unit distance apart; and likewise that the planes for w and w' are at unit distance apart. The Cartesian coordinates of any point on the z, z' line are

$$\sigma x + (1-\sigma)\, x', \quad \sigma y + (1-\sigma)\, y', \quad 1-\sigma,$$

and those of any point on the w, w' line are

$$\rho u + (1-\rho)\, u', \quad \rho v + (1-\rho)\, v', \quad 1-\rho,$$

where ρ and σ are real quantities. If, then, by means of two relations

$$F(w, w', z, z') = 0, \quad G(w, w', z, z') = 0,$$

the whole z, z' line is transformed into the whole w, w' line, there must be some rational correspondence between the variable points upon the lines; and therefore the coordinates of the one current point must be connected, by functional relations, with the coordinates of the other current point.

From the equations $F=0$ and $G=0$, it follows that u, v, u', v' are functions of x, y, x', y' only and do not involve σ; and, similarly, x, y, x', y' are functions of u, v, u', v' only and do not involve ρ. Hence ρ is a function of σ only; and it must be such that it takes the values 0 and 1 (in either order) when σ has the values 0 and 1. For the other coordinates of the point on the line, we have

$$\rho u + (1-\rho)\, u' = f(\xi, \eta, 1-\sigma),$$
$$\rho v + (1-\rho)\, v' = g(\xi, \eta, 1-\sigma),$$

where $\xi = \sigma x + (1-\sigma)\, x', \quad \eta = \sigma y + (1-\sigma)\, y',$

and f and g denote appropriate functional forms. Hence, using the fact that ρ is a function of σ alone, we have

$$\rho \frac{\partial u}{\partial x} + (1-\rho) \frac{\partial u'}{\partial x} = \sigma \frac{\partial f}{\partial \xi}, \quad \rho \frac{\partial u}{\partial x'} + (1-\rho) \frac{\partial u'}{\partial x'} = (1-\sigma) \frac{\partial f}{\partial \xi},$$

$$\rho \frac{\partial u}{\partial y} + (1-\rho) \frac{\partial u'}{\partial y} = \sigma \frac{\partial f}{\partial \eta}, \quad \rho \frac{\partial u}{\partial y'} + (1-\rho) \frac{\partial u'}{\partial y'} = (1-\sigma) \frac{\partial f}{\partial \eta};$$

and therefore

$$\left\{\rho \frac{\partial u}{\partial x} + (1-\rho) \frac{\partial u'}{\partial x}\right\} \left\{\rho \frac{\partial u}{\partial y'} + (1-\rho) \frac{\partial u'}{\partial y'}\right\}$$
$$= \left\{\rho \frac{\partial u}{\partial y} + (1-\rho) \frac{\partial u'}{\partial y}\right\} \left\{\rho \frac{\partial u}{\partial x'} + (1-\rho) \frac{\partial u'}{\partial x'}\right\}.$$

This holds for all values of ρ; and the quantities u, u' do not involve ρ; hence

$$\frac{\partial u}{\partial x} \frac{\partial u}{\partial y'} = \frac{\partial u}{\partial y} \frac{\partial u}{\partial x'},$$

$$\frac{\partial u}{\partial x} \frac{\partial u'}{\partial y'} + \frac{\partial u'}{\partial x} \frac{\partial u}{\partial y'} = \frac{\partial u}{\partial y} \frac{\partial u'}{\partial x'} + \frac{\partial u}{\partial x'} \frac{\partial u'}{\partial y},$$

$$\frac{\partial u'}{\partial x} \frac{\partial u'}{\partial y'} = \frac{\partial u'}{\partial y} \frac{\partial u'}{\partial x'}.$$

Proceeding similarly from the equation

$$\rho v + (1-\rho)\, v' = g(\xi, \eta, 1-\sigma),$$

we find

$$\frac{\partial v}{\partial x} \frac{\partial v}{\partial y'} = \frac{\partial v}{\partial y} \frac{\partial v}{\partial x'},$$

$$\frac{\partial v}{\partial x} \frac{\partial v'}{\partial y'} + \frac{\partial v'}{\partial x} \frac{\partial v}{\partial y'} = \frac{\partial v}{\partial y} \frac{\partial v'}{\partial x'} + \frac{\partial v}{\partial x'} \frac{\partial v'}{\partial y},$$

$$\frac{\partial v'}{\partial x} \frac{\partial v'}{\partial y'} = \frac{\partial v'}{\partial y} \frac{\partial v'}{\partial x'}.$$

Moreover, we always have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial x'} = \frac{\partial v}{\partial y'}, \quad \frac{\partial u}{\partial y'} = -\frac{\partial v}{\partial x'},$$

$$\frac{\partial u'}{\partial x} = \frac{\partial v'}{\partial y}, \quad \frac{\partial u'}{\partial y} = -\frac{\partial v'}{\partial x}, \quad \frac{\partial u'}{\partial x'} = \frac{\partial v'}{\partial y'}, \quad \frac{\partial u'}{\partial y'} = -\frac{\partial v'}{\partial x'}.$$

because $u+iv$ and $u'+iv'$ are functions of z and z'. These permanent relations can be used to transform the three equations involving derivatives of v and v' into the three equations involving derivatives of u and u' above: hence, as the permanent relations always are retained, we keep (for the present) only the three equations involving derivatives of u and u'.

The complete integral (save as to an additive constant) of the first of the three equations, which involves u only, is

$$u=\alpha x-\beta y+\alpha' x'-\beta' y',$$

provided the constants satisfy the relation

$$\alpha\beta'-\alpha'\beta=0;$$

and the permanent relations then give

$$v=\beta x+\alpha y+\beta' x'+\alpha' y',$$

so that $$w=u+iv=(\alpha+i\beta)\,z+(\alpha'+i\beta')\,z'.$$

Now let $\alpha+i\beta=Ae^{\mu i}$, $\alpha'+i\beta'=A'e^{\mu' i}$; then the condition

$$\alpha\beta'-\alpha'\beta=0$$

gives $$AA'\sin(\mu-\mu')=0,$$

so that either $A=0$, or $A'=0$, or $\mu=\mu'$.

Similarly the complete integral (save as to an additive constant) of the third of the three retained equations, which involves u' only, is

$$u'=\gamma x-\delta y+\gamma' x'-\delta' y',$$

provided the constants satisfy the relation

$$\gamma\delta'-\gamma'\delta=0;$$

and the permanent relations then give

$$v'=\delta x+\gamma y+\delta' x'+\gamma' y',$$

so that $$w'=u'+iv'=(\gamma+i\delta)\,z+(\gamma'+i\delta')\,z'.$$

Now let $\gamma+i\delta=Ce^{\nu i}$, $\gamma'+i\delta'=C'e^{\nu' i}$; then the condition

$$\gamma\delta'-\gamma'\delta=0$$

gives $$CC'\sin(\nu-\nu')=0,$$

so that either $C=0$, or $C'=0$, or $\nu=\nu'$.

The second of the three retained equations, which involves u and u', is satisfied identically provided

$$\alpha\delta' - \alpha'\delta = \beta\gamma' - \beta'\gamma,$$

and this condition requires the equation

$$AC' \sin(\mu - \nu') = A'C \sin(\mu' - \nu).$$

There are nine cases to be considered, viz., the three alternatives out of the first equation to be combined with the three alternatives out of the third equation, and these are subject to the condition arising out of the second equation. Moreover, w and w' must be independent functions of z and z'—a limitation that renders some of the combinations inadmissible. Thus, consider the combination $\mu = \mu'$, $\nu = \nu'$; the condition from the second equation becomes

$$(AC' - A'C)\sin(\mu - \nu) = 0.$$

Now $\quad we^{-\mu i} = Az + A'z', \quad w'e^{-\nu i} = Cz + C'z',$

and, as w and w' are independent of one another, we cannot have $AC' - A'C = 0$; hence $\mu = \nu = \mu' = \nu'$. Similarly for the other cases in detail. The aggregate of surviving effective cases is as follows:—

(i) $\quad w = (Az + A'z')\, e^{\mu i}, \quad w' = (Cz + C'z')\, e^{\mu i};$

(ii) $\quad w = (Az + A'z')\, e^{\mu i}, \quad w' = Cze^{\mu i};$

(iii) $\quad w = (Az + A'z')\, e^{\mu i}, \quad w' = C'z'e^{\mu i};$

(iv) $\quad w = Aze^{\mu i}, \qquad\qquad w' = C'z'e^{\mu i};$

(v) $\quad w = A'z'e^{\mu i}, \qquad\qquad w' = (Cz + C'z')\, e^{\mu i};$

(vi) $\quad w = Aze^{\mu i}, \qquad\qquad w' = (Cz + C'z')\, e^{\mu i};$

(vii) $\quad w = A'z'e^{\mu i}, \qquad\qquad w' = Cze^{\mu i}.$

Of these possibilities (ii), (iii), (v), (vi) effectively are the same, owing to a manifestly admissible interchange of z and z', and also of w and w'; and for a like reason (iv) and (vii) are effectively the same. Thus solely, so far as the special differential equations for u and u' are concerned, three cases have to be considered. But the condition, applying to ρ and σ as functions of one another alone, must be observed: ρ is to be 0 or 1, when σ is 0 or 1. From the earliest equations we have

$$\frac{\rho\alpha + (1-\rho)\gamma}{\sigma} = \frac{\rho\alpha' + (1-\rho)\gamma'}{1-\sigma},$$

$$\frac{\rho\beta + (1-\rho)\delta}{\sigma} = \frac{\rho\beta' + (1-\rho)\delta'}{1-\sigma},$$

and therefore, for case (i) above, we have

$$\frac{\rho A + (1-\rho)C}{\rho A' + (1-\rho)C'} = \frac{\sigma}{1-\rho}.$$

This is the relation, in the present case, between ρ and σ. If $\rho = 1$ when $\sigma = 1$, we have $A' = 0$; and then we must have $\rho = 0$ when $\sigma = 0$, that is, $C = 0$; so that the relations are

$$w = Aze^{\mu i}, \quad w' = C'z'e^{\mu i}.$$

If $\rho = 0$ when $\sigma = 1$, we have $C' = 0$; and then we must have $\rho = 1$ when $\sigma = 0$, that is, $A = 0$: so that the relations are

$$w = A'z'e^{\mu i}, \qquad w' = Cze^{\mu i},$$

effectively the same as before. And so for the other cases.

Thus we have the result:—

The only transformations, which give a rational conservation of the straight line joining z and z' in two parallel planes into a straight line joining w and w' also in two parallel planes, are

$$w = Aze^{\mu i}, \qquad w' = C'z'e^{\mu i},$$

where A, C', μ are real constants; or the transformation can be changed into this form by interchange of z and z' or by interchange of w and w'.

These transformations are so trivial as to be negligible in the general theory; and so we can assert generally that *general functional relations of the form*

$$F(w, w', z, z') = 0, \quad G(w, w', z, z') = 0,$$

while they transform a pair of points, one in the z-plane and one in the z'-plane, into a pair (or into more than a single pair) of points, one in the w-plane and one in the w'-plane, do not give a birational transformation of the straight line joining z and z' into the straight line joining w and w'.

Note I. The net inference is that the equations of relation between w, w', z, z' will transform a pair (or pairs) of points z and z' in their planes into a pair (or pairs) of points w and w' also in their planes. To this result we shall return.

Note II. The preceding investigation suggests a cognate question, which will only be propounded, as follows:

Two functional relations, $F=0$ and $G=0$, between w, w', z, z' transform a pair of points z and z', in parallel planes, into a pair of points w and w', also in parallel planes: let z and z' be connected by any analytic curve. Then w and w' can be connected by an analytic curve (or by more than one analytic curve) having a birational correspondence with the curve joining z and z'. How are the equations of this correspondence connected (if at all) with the original functional relations when the z-z' curve and the w-w' curve are assigned? What are the equations of the w-w' curve when functional relations are given and the z-z' curve is assigned, it being always assumed that the correspondence between the curves is birational?

6. It thus appears that, if a straight line is taken to represent a couple of complex variables and is constituted in the manner explained, the only effective part of the representation is provided by the two points in which the line cuts the parallel planes; the variables of these two points are themselves the actual complex variables in question.

A straight line in ordinary space, for the precise determination of which four real variables always are required, might be selected in another fashion. It would be made unique by the point in which it cuts a given plane of reference and by its position relative to fixed axes of reference. Thus, for example, if the coordinates relative to axes in the plane were x and y, we should take

$$z=x+iy,$$

and if the angular coordinates of the line were $\tan\theta$, where θ is its inclination to a normal to the plane, and ϕ, the orientation of a plane through this normal and the line, we should take

$$z'=e^{i\phi}\tan\theta.$$

It is easy to see that a geometrical representation of the variables z and z' by a straight line thus selected is not in the least more comprehensive than was the former representation; for, when we take a plane parallel to, and at unit distance from, the plane of reference, the complex coordinate of the point, where the line cuts the new plane, is z'', where

$$z''=z+z'.$$

All the earlier investigation applies when z and z'' are taken as the complex variables. The whole straight line is not a representation of the two variables, and the only representation that is provided arises from the complex variable of the point, where it cuts the plane, and the complex variable of its direction at the point.

Still another method of constructing a straight line to represent the variables z and z' has been propounded by Vivanti,* whereby it is given as the intersection of the two planes

$$xX + yZ = 1, \quad x'Y + y'Z = 1.$$

The vicinity of a line z_0, z_0' is assumed to be the aggregate of all lines such that

$$(x - x_0)^2 + (y - y_0)^2 \leqq \rho^2, \quad (x' - x_0')^2 + (y' - y_0')^2 \leqq \sigma^2,$$

where ρ and σ are small arbitrary quantities, and the boundary of the vicinity is made up of the double complex of lines

$$(x - x_0)^2 + (y - y_0)^2 = \rho^2, \quad (x' - x_0')^2 + (y' - y_0')^2 = \sigma^2.$$

As before, it is easy to see that the whole straight line, as a single entity, is not a representation of the two complex variables z and z'; the representation provided depends upon the coordinates of the two points in which the line cuts the plane of X, Y and the plane of X, Z (or any two of the coordinate planes).

Note. The preceding remarks refer to the possibility of representing two complex variables by a straight line joining two points: and they could be extended so as to deal with an analytical curve joining the two points.

They do not refer to functions of two complex variables which are functions of a line. Functions of a real line occur in mathematical physics, *e.g.*, the energy of a current in a wire; and this notion has been extended by Volterra,† on the basis of Poincaré's generalisation of one of Cauchy's integral-theorems. In the case of the integral of a uniform function of a single complex variable ranging from an assigned point, it is known that the integral is zero round a contour enclosing no singularity and that the integral taken up to any

* *Rend. Circ. Mat. Palermo*, t. ix. (1895), pp. 108–121.
† *Acta Math.*, t. xii. (1889), pp. 233–286.

other point is (subject to the usual proviso as to singularities) independent of the path between the points, that is, the integral is a function of the final point. So also the integral of a function of two complex variables over a closed surface in four-dimensional space is zero if the surface encloses no singularity of the function: while, if the surface is not closed, the integral (subject to a similar proviso as to singularities) depends upon the boundary of the surface, that is, the integral is a function of a line.

Such an issue is completely distinct from the line-representation at present under discussion.

7. We thus fall back upon the simple, but not very suggestive, method of representing the two variables z and z' by two points, either in one and the same plane, or in different planes, the two points always being unrelated. It is the method most generally adopted by Picard* and others. It is useful for quite simple purposes; thus it is employed by Picard in dealing with the residues of the double integrals of rational functions, and it is important in his theory of the periods of double integrals of algebraical functions. It is a known theorem, due originally to Jacobi, that a uniform function of two variables cannot have more than four pairs of periods. The geometrical representation of two independent variables by means of two unrelated points admits of a simple presentation of single periodicity, of double periodicity, and (as will be indicated elsewhere) of triple periodicity: but it does not lend itself to the representation of quadruple periodicity, such as characterises the quotients of the double theta-functions.

It is unnecessary to insist on the limitations and the deficiencies of the representation by points. One deficiency has been indicated, in connection with quadruple periodicity: one limitation will be indicated almost immediately, in connection with equations of transformation.

8. A few simple inferences from the geometrical representation by points can immediately be made. Take any joint "place" made up of z and z', two points in the planes, and associate with it a joint place made up of the dependent variables by w and w', two points in new planes, it being

* In his *Traité d'Analyse*, his *Théorie des fonctions algébriques* (already quoted), and his memoirs on the latter subject, *passim*.

assumed that w and w' are two distinct functions of the independent variables, so that the Jacobian

$$J\left(\frac{w,\ w'}{z,\ z'}\right)$$

does not vanish identically. A small variation from the former place, represented by dz and dz', determines a small variation from the latter place, represented (say) by dw and dw', then

$$dw = Adz + Bdz', \quad dw' = Cdz + Ddz',$$

where $AD - BC$, being the above Jacobian, does not vanish. Again, let d_1z and d_1z', d_2z and d_2z' be a couple of small variations from the z-z' place, determining d_1w and d_1w', d_2w and d_2w' a couple of small variations from the w-w' place; then

$$\begin{vmatrix} d_1w,\ d_1w' \\ d_2w,\ d_2w' \end{vmatrix} = \begin{vmatrix} Ad_1z + Bd_1z',\ Cd_1z + Dd_1z' \\ Ad_2z + Bd_2z',\ Cd_2z + Dd_2z' \end{vmatrix}$$

$$= J \begin{vmatrix} d_1z,\ d_1z' \\ d_2z,\ d_2z' \end{vmatrix}.$$

Manifestly, if $d_1zd_2z' - d_2zd_1z'$ vanishes, then $d_1wd_2w' - d_2wd_1w'$ vanishes; and the converse holds because J is not zero. Hence, *if, at z and at z' in their respective planes, two similar infinitesimal triangles are taken, the corresponding infinitesimal triangles at w and at w' in their respective planes also are similar, and conversely.*

This property holds for all pairs of similar infinitesimal triangles at the point, and therefore, when at respective points, the z-plane and the z'-plane are put into conformal relation, the w-plane and the w'-plane, at the corresponding point, are also put into conformal relation. If the whole z-piane is put into conformal relation with the whole z'-plane, then the whole w-plane is put into conformal relation with the whole w'-plane: a geometrical method of stating an obvious analytical result that the equations

$$F(w, w', z, z') = 0, \quad G(w, w', z, z') = 0, \quad \phi(z, z') = 0,$$

lead to an equation

$$\psi(w, w') = 0.$$

Another interpretation of the relation

$$\begin{vmatrix} d_1w,\ d_1w' \\ d_2w,\ d_2w' \end{vmatrix} = J \begin{vmatrix} d_1z,\ d_1z' \\ d_2z,\ d_2z' \end{vmatrix}$$

is as follows: *When w and w' are two independent complex functions of two independent complex variables z and z', and when d_1z, d_1z', d_1w, d_1w' are one set of simultaneous small variations, while d_2z, d_2z', d_2w, d_2w' are another set of simultaneous small variations, the quantity*

$$\begin{vmatrix} d_1w, & d_1w' \\ d_2w, & d_2w' \end{vmatrix} \div \begin{vmatrix} d_1z, & d_1z' \\ d_2z, & d_2z' \end{vmatrix}$$

is independent of differential elements, whatever these may be.

9. The converse of the latter also is true, viz.:—
Let z and z' be two complex variables, such that

$$z = x + iy, \qquad z' = x' + iy',$$

where x, y, x', y' are four real and independent variables; and let w and w' be two complex quantities, distinct from one another, say

$$w = u + iv, \qquad w' = u' + iv',$$

where u, v, u', v' are real quantities, being functions of x, y, x', y'; then when the magnitude

$$\begin{vmatrix} d_1w, & d_1w' \\ d_2w, & d_2w' \end{vmatrix} \div \begin{vmatrix} d_1z, & d_1z' \\ d_2z, & d_2z' \end{vmatrix},$$

for all infinitesimal variations, is independent of these variations, both w and w' are functions of z and z' above.

The property manifestly, for two independent complex variables, corresponds to Riemann's definition-property for functionality in the case of a single complex variable: it can be established as follows. Let

$$\frac{\partial w}{\partial x} = \alpha, \quad \frac{\partial w}{\partial y} = \beta, \quad \frac{\partial w}{\partial x'} = \gamma, \quad \frac{\partial w}{\partial y'} = \delta,$$

$$\frac{\partial w'}{\partial x} = \alpha', \quad \frac{\partial w'}{\partial y} = \beta', \quad \frac{\partial w'}{\partial x'} = \gamma', \quad \frac{\partial w'}{\partial y'} = \delta',$$

Then

$$\begin{vmatrix} d_1w, & d_1w' \\ d_2w, & d_2w' \end{vmatrix}$$

$$= \begin{vmatrix} \alpha d_1x + \beta d_1y + \gamma d_1x' + \delta d_1y', & \alpha' d_1x + \beta' d_1y + \gamma' d_1x' + \delta' d_1y' \\ \alpha d_2x + \beta d_2y + \gamma d_2x' + \delta d_2y', & \alpha' d_2x + \beta' d_2y + \gamma' d_2x' + \delta' d_2y \end{vmatrix}$$

$$= \begin{vmatrix} \alpha, & \alpha' \\ \beta, & \beta' \end{vmatrix} \begin{vmatrix} d_1x, & d_1y \\ d_2x, & d_2y \end{vmatrix} + \begin{vmatrix} \alpha, & \alpha' \\ \gamma, & \gamma' \end{vmatrix} \begin{vmatrix} d_1x, & d_1x' \\ d_2x, & d_2x' \end{vmatrix} + \begin{vmatrix} \alpha, & \alpha' \\ \delta, & \delta' \end{vmatrix} \begin{vmatrix} d_1x, & d_1y' \\ d_2x, & d_2y' \end{vmatrix}$$

$$+ \begin{vmatrix} \beta, & \beta' \\ \gamma, & \gamma' \end{vmatrix} \begin{vmatrix} d_1y, & d_1x' \\ d_2y, & d_2x' \end{vmatrix} + \begin{vmatrix} \beta, & \beta' \\ \delta, & \delta' \end{vmatrix} \begin{vmatrix} d_1y, & d_1y' \\ d_2y, & d_2y' \end{vmatrix} + \begin{vmatrix} \gamma, & \gamma' \\ \delta, & \delta' \end{vmatrix} \begin{vmatrix} d_1x', & d_1y' \\ d_2x', & d_2y' \end{vmatrix};$$

and

$$\begin{vmatrix} d_1z, & d_1z' \\ d_2z, & d_2z' \end{vmatrix} = \begin{vmatrix} d_1x + id_1y, & d_1x' + id_1y' \\ d_2x + id_2y, & d_2x' + id_2y' \end{vmatrix}$$

$$= \begin{vmatrix} d_1x, & d_1x' \\ d_2x, & d_2x' \end{vmatrix} + i \begin{vmatrix} d_1y, & d_1x' \\ d_2y, & d_2x' \end{vmatrix} + i \begin{vmatrix} d_1x, & d_1y' \\ d_2x, & d_2y' \end{vmatrix} - \begin{vmatrix} d_1y, & d_1y' \\ d_2y, & d_2y' \end{vmatrix}.$$

These two quantities are to have a non-vanishing ratio which is independent of the differential elements that occur; consequently, denoting the ratio by J, we must have

$$\alpha\beta' - \alpha'\beta = 0,$$
$$\alpha\gamma' - \alpha'\gamma = J,$$
$$\alpha\delta' - \alpha'\delta = iJ,$$
$$\beta\gamma' - \beta'\gamma = iJ,$$
$$\beta\delta' - \beta'\delta = -J,$$
$$\gamma\delta' - \gamma'\delta = 0,$$

as the necessary and sufficient conditions.

From the first of these equations it follows that some quantity θ exists such that

$$\frac{\partial w}{\partial y} = \beta = \theta\alpha = \theta\frac{\partial w}{\partial x}, \quad \frac{\partial w'}{\partial y} = \beta' = \theta\alpha' = \theta\frac{\partial w'}{\partial x};$$

and from the sixth of them it follows that some quantity ϕ exists such that

$$\frac{\partial w}{\partial y'} = \delta = \phi\gamma = \phi\frac{\partial w}{\partial x'}, \quad \frac{\partial w'}{\partial y'} = \delta' = \phi\gamma' = \phi\frac{\partial w'}{\partial x'}.$$

The third equation then gives

$$iJ = \alpha\delta' - \alpha'\delta = \phi(\alpha\gamma' - \alpha'\gamma) = \phi J;$$

the fourth gives

$$iJ = \beta\gamma' - \beta'\gamma = \theta(\alpha\gamma' - \alpha'\gamma) = \theta J;$$

the fifth gives

$$-J = \beta\delta' - \beta'\delta = \theta\phi(\alpha\gamma' - \alpha'\gamma) = \theta\phi J;$$

and the second gives the value of J. All the conditions are satisfied if

$$\theta = i, \quad \phi = i, \quad J = \frac{\partial w}{\partial x}\frac{\partial w'}{\partial x'} - \frac{\partial w}{\partial x'}\frac{\partial w'}{\partial x}.$$

But now

$$\frac{\partial w}{\partial y} = i\frac{\partial w}{\partial x}, \qquad \frac{\partial w}{\partial y'} = i\frac{\partial w}{\partial x'},$$

and these are the only equations affecting w alone; hence w is a function of z and z' only. Similarly

$$\frac{\partial w'}{\partial y} = i\frac{\partial w'}{\partial x}, \qquad \frac{\partial w'}{\partial y'} = i\frac{\partial w'}{\partial x'},$$

and these are the only equations affecting w' alone; hence w' is a function of z and z' alone. The value of J now is

$$J\left(\frac{w, w'}{z, z'}\right),$$

and this does not vanish identically; hence w and w' are independent functions of z and z' alone—which is the result to be established.

It will be noticed that the result has been obtained without any special consideration of the complex expression $u + iv$ for w and the complex expression $u' + iv'$ for w'. When these are introduced, each of the six equations produces two equations in real magnitudes only: and denoting J by $P + iQ$, where P and Q are real, we find the twelve equations in the form

$$\left(\frac{u, u'}{x, y}\right) - \left(\frac{v, v'}{x, y}\right) = O, \quad \left(\frac{u, v'}{x, y}\right) + \left(\frac{v, u'}{x, y}\right) = O;$$

$$\left(\frac{u, u'}{x, x'}\right) - \left(\frac{v, v'}{x, x'}\right) = P, \quad \left(\frac{u, v'}{x, x'}\right) + \left(\frac{v, u'}{x, x'}\right) = Q;$$

$$\left(\frac{u, u'}{y, x'}\right) - \left(\frac{v, v'}{y, x'}\right) = -Q, \quad \left(\frac{u, v'}{y, x'}\right) + \left(\frac{v, u'}{y, x'}\right) = P;$$

$$\left(\frac{u, u'}{x, y'}\right) - \left(\frac{v, v'}{x, y'}\right) = -Q, \quad \left(\frac{u, v'}{x, y'}\right) + \left(\frac{v, u'}{y, x'}\right) = P;$$

$$\left(\frac{u, u'}{y, y'}\right) - \left(\frac{v, v'}{y, y'}\right) = -P, \quad \left(\frac{u, v'}{y, y'}\right) + \left(\frac{v, u'}{y, y'}\right) = -Q;$$

$$\left(\frac{u, u'}{x', y'}\right) - \left(\frac{v, v'}{x', y'}\right) = O, \quad \left(\frac{u, v'}{x', y'}\right) + \left(\frac{v, u'}{x', y'}\right) = O;$$

where, throughout,

$$\left(\frac{U, V}{X, Y}\right) \text{ denotes } \frac{\partial U}{\partial X}\frac{\partial V}{\partial Y} - \frac{\partial U}{\partial Y}\frac{\partial V}{\partial X}$$

for all combinations of the quantities.

10. The conformation property, established in association with functions of a single variable and used to interpret a relation

$$\frac{\delta w}{\delta' w} = \frac{\delta z}{\delta' z},$$

suggests a corresponding interpretation* in connection with the quantity

$$\begin{vmatrix} d_1 z, & d_1 z' \\ d_2 z, & d_2 z' \end{vmatrix} = d_1 z d_2 z' - d_2 z d_1 z'.$$

For simplicity let z and z' be represented in the same plane At any point O in the plane take OA, OB, OC, OD to represent $d_1 z$, $d_1 z'$, $d_2 z$, $d_2 z'$. Along the bisector of the angle between OA and OD take OP a mean proportional between OA and OD, and along the bisector of the angle between OB and OC take OQ a mean proportional between OB and OC. Complete the parallelogram of which OP and OQ are adjacent sides: let M be the product of its diagonals, and Θ the sum of the inclinations of these diagonals to the axis of real quantities; then

$$d_1 z d_2 z' - d_2 z d_1 z' = M e^{\Theta i}.$$

Constructing a similar parallelogram in connection with the variations of w and w', we should have

$$d_1 w d_2 w' - d_2 w d_1 w' = N e^{\Phi i}.$$

Consequently

$$N e^{\Phi i} = J M e^{\Theta i}.$$

Now let two sets of pairs of variations of z and z', one of them leading to a quantity $M e^{\Theta i}$, the other to a quantity $M' e^{\Theta' i}$, and let the corresponding quantities, arising out of the variations of w and w', be $N e^{\Phi i}$, $N' e^{\Phi' i}$. Then we have

$$N e^{\Phi i} = J M e^{\Theta i}, \quad N' e^{\Phi' i} = J M' e^{\Theta' i},$$

and therefore

$$\frac{N'}{N} = \frac{M'}{M}, \quad \Phi' - \Phi = \Theta' - \Theta,$$

being an extension to functions of two variables of the conformation property arising out of functions of one variable. Also the extension is determinate; for the parallelogram constructed in connection with $d_1 z d_2 z' - d_2 z d_1 z'$ is unique in magnitude and orientation.

* *Quart Jour. Math.*, vol. XLII. (1910), p. 8.

11. An extension of the conformation property for single variables can be given in a different form. When $z+\delta z$ and $z+\delta' z$ are different points adjacent to z in its plane, a quantity μ exists such that

$$\delta' z = \mu \delta z,$$

and, if $w+\delta w$ and $w+\delta' w$ be the corresponding points adjacent to the corresponding point w in its plane, then

$$\delta' w = \mu \delta w.$$

The conformation property can be expressed by saying that, in the transformation from the z-plane to the w-plane, *the complex multiplier μ is unaltered.*

Now let $z+d_1 z,\ z'+d_1 z'$; $z+d_2 z,\ z'+d_2 z'$; $z+d_3 z,\ z'+d_3 z'$ be three different places adjacent to the place $z,\ z'$; then complex quantities λ and μ exist such that

$$\left.\begin{aligned} d_3 z &= \lambda d_1 z + \mu d_2 z \\ d_3 z' &= \lambda d_1 z' + \mu d_2 z' \end{aligned}\right\}.$$

Corresponding to these, let $w+d_1 w,\ w'+d_1 w'$; $w+d_2 w,\ w'+d_2 w'$; $w+d_3 w,\ w'+d_3 w'$ be the three places adjacent to the corresponding places $w,\ w'$; then

$$\begin{aligned} d_3 w &= A d_3 z + B d_3 z' \\ &= \lambda (A d_1 z + B d_1 z') + \mu (A d_2 z + B d_2 z') \\ &= \lambda d_1 w + \mu d_2 w, \\ d_3 w' &= \lambda d_1 w' + \mu d_2 w' : \end{aligned}$$

and these relations between the original variables $z,\ z'$ and the deduced variables $w,\ w'$ can be expressed by saying that *the complex multipliers λ and μ are unaltered.*

12. Yet while geometrical interpretations of functionality can thus be provided at any "place" in the two planes, the geometrical interpretations connected with the transformatoin of loci in the two planes that arises through functional equations are more complicated than in the case of a single variable, and do not furnish a natural extension of the simpler theory. Thus consider the periodic substitution

$$z\sqrt{2} = w + w', \qquad z'\sqrt{2} = w - w',$$

leading to

$$w\sqrt{2} = z + z', \qquad w'\sqrt{2} = z - z'.$$

Corresponding to any place made up of a point-pair in the z-plane and the z'-plane, there exists a place made up of a point-pair in the w-plane and the w'-plane. But the combination of a definite locus in the z-plane, unaffected by variations of z', and a definite locus in the z'-plane, unaffected by variations of z, does not lead to loci in the planes of w and w'. By imposing limitations upon the description of the loci in the planes of z and z' (for instance, by making z and z' functions of a single current parameter), it would be possible to obtain loci in the w-plane and the w'-plane connected with one another: but the imposition of such a limitation as is indicated in the illustration ultimately makes the functions degenerate into functions of a single variable.

With the periodic substitution suggested, suppose that z and z' describe circles in their respective planes given by

$$z = ae^{i\theta}, \qquad z' = a'e^{i\theta'};$$

then, with the usual notation, we have

$$u\sqrt{2} = a\cos\theta + a'\cos\theta', \qquad u'\sqrt{2} = a\cos\theta - a'\cos\theta',$$
$$v\sqrt{2} = a\sin\theta + a'\sin\theta', \qquad v'\sqrt{2} = a\sin\theta - a'\sin\theta'.$$

Consequently, pairs of equations valid in the w- and w'-planes are

$$\left.\begin{aligned} 2a^2 &= (u+u')^2 + (v+v')^2 \\ 2c^2 &= (u-u')^2 + (v-v')^2 \end{aligned}\right\};$$

$$\left.\begin{aligned} 2v &= \{2a^2 - (u+u')^2\}^{\frac{1}{2}} + \{2c^2 - (u-u')^2\}^{\frac{1}{2}} \\ 2v' &= \{2a^2 - (u+u')^2\}^{\frac{1}{2}} - \{2c^2 - (u-u')^2\}^{\frac{1}{2}} \end{aligned}\right\};$$

and so on. The z-circle and the z'-circle, taken together but left unrelated in their descriptions, determine ∞^2 point-pairs in the w- and w'-planes: but there is no locus either in the w-plane or in the w'-plane thus provided through the transformation.

If, however, related loci in the two planes of z and z' are taken, it may be possible to obtain related loci in the two planes of w and w'. Thus if the z'-curve is made to depend upon the z-curve by a relation $z' = az + b$, where a and b are constants, then a w-locus independent of w' is given by

$$w\sqrt{2} = (1+a)z + b,$$

and a w'-locus independent of w is given by

$$w'\sqrt{2} = (1-a)z - b.$$

Each of the quantities w and w' is temporarily made a function of one variable only.

13. The results, that arise from the application even of the foregoing simple periodic substitution, and other considerations also, suggest that, instead of taking initially a couple of unrelated loci in the planes of z and z', it is desirable to take the geometrical content of two equations

$$\phi(x, y, x', y') = 0, \quad \psi(x, y, x', y') = 0.$$

There is at least the analytical advantage that, when the variables are changed from z and z' to w and w' by equations

$$F(w, w', z, z') = 0, \quad G(w, w', z, z') = 0,$$

the limiting equations in the planes of w and w' are of the same type, being of the form

$$\Phi(u, v, u', v') = 0, \quad \Psi(u, v, u', v') = 0.$$

But then it becomes necessary to have some method of dealing with the equations, so as to give some indication of their geometrical content.

One such method is as follows: by taking account of the fact that $\phi = 0$ and $\psi = 0$ virtually express any two of the four real quantities x, y, x', y' in terms of the other two, it makes provision for the expression of all four of them in terms of two real current paremeters. Let

$$x, y, x', y' = a, b, a', b'$$

be an ordinary simple place for the equations, say such that no one of the first derivatives of ϕ and of ψ vanishes for those values; and in its vicinity let

$$x = a + \xi, \quad y = b + \eta, \quad x' = a' + \xi', \quad y' = b' + \eta'.$$

Then we have

$$0 = \xi \frac{\partial \phi}{\partial a} + \eta \frac{\partial \phi}{\partial b} + \xi' \frac{\partial \phi}{\partial a'} + \eta' \frac{\partial \phi}{\partial b'} + [\xi, \eta, \xi', \eta']_2 + \ldots,$$

$$0 = \xi \frac{\partial \psi}{\partial a} + \eta \frac{\partial \psi}{\partial b} + \xi' \frac{\partial \psi}{\partial a'} + \eta' \frac{\partial \psi}{\partial b'} + [\xi, \eta, \xi', \eta']_2 + \ldots.$$

Introduce two new parameters s and t, and take

$$\left.\begin{aligned} s &= \xi\alpha + \eta\beta + \xi'\gamma + \eta'\delta \\ t &= \xi\alpha' + \eta\beta' + \xi'\gamma' + \eta'\delta' \end{aligned}\right\},$$

where α, β, γ, δ, α', β', γ', δ' are constants, such that the determinant

$$\begin{vmatrix} \frac{\partial\phi}{\partial a}, & \frac{\partial\phi}{\partial b}, & \frac{\partial\phi}{\partial c}, & \frac{\partial\phi}{\partial d} \\ \frac{\partial\psi}{\partial a}, & \frac{\partial\psi}{\partial b}, & \frac{\partial\psi}{\partial c}, & \frac{\partial\psi}{\partial d} \\ \alpha\ , & \beta\ , & \gamma\ , & \delta \\ \alpha', & \beta', & \gamma', & \delta' \end{vmatrix}$$

does not vanish. Then the four equations can be resolved so as to express ξ, η, ξ', η' in terms of s and t; owing to the limitations imposed, it happens that the expressions are regular functions of s and t, vanishing with them; and so we have

$$x, y, x', y' = \text{functions of two real parameters } s \text{ and } t.$$

It then is sufficient to consider regions of variation in the plane of the variables s and t.

Corresponding to a region in the s-t plane, bounded by a curve $k(s, t) = 0$, we have ranges of variation in the z and z' planes limited by

$$\left.\begin{aligned} x - a &= p(s, t) \\ y - b &= q(s, t) \\ 0 &= k(s, t) \end{aligned}\right\}, \qquad \left.\begin{aligned} x' - a' &= p'(s, t) \\ y' - b' &= q'(s, t) \\ 0 &= k(s, t) \end{aligned}\right\},$$

that is, limited by curves

$$f(x - a,\ y - b) = 0, \qquad g(x' - a',\ y' - b') = 0,$$

the current descriptions of which are related. Moreover, from the equations $F = 0$ and $G = 0$, we have u, v, u', v' potentially expressible in terms of x, y, x', y'; and so the whole range of the variations in the w and w' planes would be given by taking these expressions and substituting the values of x, y, x', y' in terms of s and t, that is, the range of variation is connected with equations

$$u, v, u', v' = \text{functions of two real parameters } s \text{ and } t.$$

But, in dealing with the geometrical content in either set of planes of reference, care must be exercised as to what is justly included. Thus, in the example indicated, we are not to include every point within the curve $f = 0$ conjointly with every point within the curve $g = 0$; we can include each

point within the curve $f=0$ conjointly with the point in $g=0$ appropriately associated with it through the values of s and t.

14. The method given for the expression of x, y, x', y' is general in form: but there is no necessity to adopt it when simpler processes of expression can be employed. Thus in the case of the equations

$$x^2+y^2+x'^2=1, \qquad x^2-y^2=y',$$

a (completely periodic) representation of the variables is

$$\left.\begin{aligned} x&=\sin s\cos t\\ y&=\sin s\sin t\end{aligned}\right\}, \qquad \left.\begin{aligned} x'&=\cos s\\ y'&=\sin^2 s\cos 2t\end{aligned}\right\};$$

if the range in the plane of s and t be the area of the triangle bounded by the lines

$$s-t=0, \qquad s+t=\tfrac{1}{2}\pi, \qquad t=0,$$

the area in the z-plane is a curvilinear figure bounded by a straight line and two quarter-circles, while the area in the z'-plane is bounded by a parabola and by arcs of the two curves

$$y'=(1-x'^2)\,(2x'^2-1), \qquad y'=-(1-x'^2)\,(2x'^2-1).$$

Again, for the periodic substitution

$$w\sqrt{2}=z+z', \qquad w'\sqrt{2}=z-z',$$

a range in the z- and z'-planes limited by

$$x^2+x'^2=1, \qquad y^2+y'^2=1,$$

gives a range in the w- and w'-planes limited by

$$u^2+u'^2=1, \qquad v^2+v'^2=1.$$

Similar examples to any number can be constructed to illustrate the preceding general remarks about the relations of the two sets of planes and the ranges of variation in each set.

24th August, 1910.

ON IRREGULAR DETERMINANTS.

By *Thorold Gosset.*

Cayley has tabulated the composition of the classes of binary quadratic forms for the irregular negative determinants which are numerically less than 1000 (*Crelle's Journal*, vol. lx., p. 370).

Thirteen negative determinants are there given as irregular:—243, 307, 339, 459, 576, 580, 675, 755, 820, 884, 891, 900, 974.

Perott subsequently discovered two other irregular negative determinants less than 1000:—468 and 931 (*Crelle's Journal*, vol. xcv., p. 236).

I find that there still remain three more to be added to the list—544, 547, and 972. They are given incorrectly as regular in Gauss' tables. The generic characters and the composition of the different classes of these three determinants are shown in the following tables.

$D = -544 = -2^5.17$. Four genera. Sixteen classes.

Classes.	$\left(\frac{-1}{n}\right)$	$\left(\frac{2}{n}\right)$	$\left(\frac{n}{17}\right)$	Composition.
(1, 0, 544)	+	+	+	$f^4 = g^4 = 1$
(4, 2, 137)	+	+	+	f^2
(17, 0, 32)	+	+	+	g^2
(25, 9, 25)	+	+	+	$f^2 g^2$
(16, 4, 35)	−	−	+	f
(16, −4, 35)	−	−	+	f^3
(19, 8, 32)	−	−	+	fg^2
(19, −8, 32)	−	−	+	$f^3 g^3$
(7, 3, 79)	−	+	−	g
(7, −3, 79)	−	+	−	g^3
(23, −10, 28)	−	+	−	$f^2 g$
(23, 10, 28)	−	+	−	$f^2 g^3$

Classes.	$\left(\frac{-1}{n}\right)$	$\left(\frac{2}{n}\right)$	$\left(\frac{n}{17}\right)$	Composition.
(20, −6, 29)	+	−	−	fg
(20, 6, 29)	+	−	−	f^3g^3
(5, 1, 109)	+	−	−	fg^3
(5, −1, 109)	+	−	−	f^3g

$D=-547$ (prime). One genus. Nine classes.

Classes.	Composition.
(1, 0, 547)	$f^3=g^3=1$
(4, 1, 137)	f
(4, −1, 137)	f^2
(11, 5, 52)	g
(11, −5, 52)	g^2
(13, −5, 44)	fg
(13, 5, 44)	f^2g^2
(19, 2, 29)	fg^2
(19, −2, 29)	f^2g

$D=-972=-2^2.3^5$. Two genera. Eighteen classes.

Classes.	$\left(\frac{-1}{n}\right)$	$\left(\frac{n}{3}\right)$	Composition.
(1, 0, 972)	+	+	$f^3=g^6=1$
(16, 2, 61)	+	+	f
(16, −2, 61)	+	+	f^2
(9, 3, 109)	+	+	g^2
(9, −3, 109)	+	+	g^4
(28, −8, 37)	+	+	fg^2
(28, 8, 37)	+	+	f^2g^4
(13, 4, 76)	+	+	fg^4
(13, −4, 76)	+	+	f^2g^2

Classes.	$\left(\frac{-1}{n}\right)$	$\left(\frac{n}{3}\right)$	Composition.
(4, 0, 243)	−	+	g^3
(16, −6, 63)	−	+	fg^3
(16, 6, 63)	−	+	f^2g^3
(31, 12, 36)	−	+	g
(31, −12, 36)	−	+	g^5
(19, −4, 52)	−	+	fg
(19, 4, 52)	−	+	f^2g^5
(7, −1, 139)	−	+	fg^5
(7, 1, 139)	−	+	f^2g

THE GRAVITATIONAL POTENTIAL AND ENERGY OF HARMONIC DEFORMATIONS OF ANY ORDER.

By *N. Mudd.*

1. The object of the following communication is to explain a method, which I have had occasion to use recently, for obtaining the potential of a harmonic deformation on a surface to any required degree of accuracy. It is usually possible to write down the potential of the surface when we can neglect the squares and higher powers of the deformation, *i.e.*, when the deformation can be treated as surface density. The squares of the small quantities occurring in the definition of the deformed surface can be included by the help of an elegant method* devised by M. Poincaré, but hitherto, so far as I am aware, no method has been in general use for carrying the approximation to any higher order.

In the last part of the paper the method is applied to determine to the fourth order the value of $\omega^2/2\pi\rho$ in terms of the ellipticity in the case of Maclaurin's ellipsoid. It further

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* *Phil. Trans.*, vol. 198, ser. *A*.

appears that the method leads to the accurate value for this expression in finite terms.

2. Let the surface of a gravitating body, called the standard surface S_0, be expressible in the form $\xi = \alpha$ in a system of orthogonal curvilinear coordinates ξ, η, ζ. Let its potential for given density be written $F_1(\xi, \eta, \zeta)$ at internal and $F_2(\xi, \eta, \zeta)$ at external points.

Consider the deformed surface $\xi = \alpha + \delta\xi$, where $\delta\xi$ is a function of η, ζ, and small constants. We suppose that the potential of a general deformation is known when this can be treated as surface density. Let P be a point outside both S_0 and the deformed surface, Q a point inside both, and denote by R the distance from P or Q of an element of the deformation considered as either positive or negative density as the case may be.

The accurate expression for the potential at P or Q is

$$\iiint \frac{\rho\, dv}{R} = \iiint \frac{\rho J(\xi, \eta, \zeta)}{R}\, d\xi\, d\eta\, d\zeta$$

(taken through the deformation)

$$= \iint_{\text{over surface}} d\eta\, d\zeta \int_{\alpha}^{\alpha+\delta\xi} \frac{\rho J(\xi, \eta, \zeta)}{R}\, d\xi$$

$$= \iint d\eta\, d\zeta \left\{ \frac{\rho J(\alpha, \eta, \zeta)}{R(\alpha, \eta, \zeta)}\, \delta\xi + \rho \frac{\delta\xi^2}{2!} \frac{\partial}{\partial\alpha} \frac{J(\alpha)}{R(\alpha)} + \rho \frac{\delta\xi^3}{3!} \frac{\partial^2}{\partial\alpha^2} \frac{J(\alpha)}{R(\alpha)} + \cdots \right\}$$

on expanding the integral as a Taylor's series as long as this is permissible. If the last term in the Taylor's series is the n^{th}, $\rho \frac{\delta\xi^n}{n!} \left(\frac{\partial^n}{\partial\alpha^n} \frac{J}{R} \right)_{\alpha+\varsigma\delta\xi}$, then the potential, correct to the $n-1^{\text{th}}$ powers of the small quantities, is

$$\iint d\zeta\, d\eta \left\{ \frac{\rho J}{R}\, \delta\xi + \rho \frac{\delta\xi^2}{2!} \frac{\partial}{\partial\alpha} \frac{J}{R} + \cdots + \rho \frac{\delta\xi^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial\alpha^{n-1}} \frac{J}{R} \right\}.$$

Now, if the limits of η and ζ on the surface are simple numbers not involving α (*e.g.*, 0 and 2π), it will be legitimate to bring the differentiations outside the double integral, and

we shall obtain for the potential of the deformation the following:—

Potential at P or Q of $\delta\xi$ considered as surface density

$+\frac{1}{2!}\frac{\partial}{\partial\alpha}$ {potential at P or Q of $\delta\xi^2$ considered as surface density},

.....

$+\frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}$

{potential at P or Q of $\delta\xi^{n-1}$ considered as surface density}.

Now, if the potential of a general deformation is known when treated as surface density, the potential of the deformation $d\xi=\delta\xi^m$ is known. Hence the potential at P or Q is found to the $(n-1)^{\text{th}}$ order, where the integral considered can be developed as far as the n^{th} term in a Taylor's series. In many important cases n may be taken as large as we please. If, now, we add respectively the functions $F_1(\xi, \eta, \zeta)$ and $F_2(\xi, \eta, \zeta)$, we obtain the potential in the following regions:—

1. Inside both S_0 and the deformed surface.

2. Outside both S_0 and the deformed surface.

But, since the result cannot depend on the standard surface chosen, it follows that the formula 1 will give the potential at all points inside the deformed surface and 2 will give the potential at all points outside the deformed surface. This consideration does not claim to be a rigorous proof of the truth of the result in the general case. The restrictions to be applied and the determination of the order to which the result is true in the general case are complicated, and, as they do not appear to be in any way illuminating, I have not thought it worth while to formulate them. The chief application is likely to be to quadric surfaces, and the consideration of any special case which presents itself is not likely to be difficult.

As an illustration of the result, consider the case when the deformed surface is $r=a+h$ (constant), $r=a$ being the standard surface. The formula gives, for the potential inside the deformed figure,

$$\frac{2\pi\rho}{3}(3a^2-r^2)+4\pi\rho\, ah+\frac{h^2}{2!}4\pi\rho=\frac{2\pi\rho}{3}\{3(a+h)^2-r^2\},$$

and for outside points

$$\frac{4\pi\rho}{3}\frac{a^3}{r}+\frac{4\pi\rho\, a^2 h}{r}+\frac{8\pi\rho\, h^2 a}{2!\,r}+\frac{8\pi\rho\, h^2}{3!\,r}=\frac{4\pi\rho}{3r}(a+h)^3,$$

which are clearly true.

3. Let us now consider the energy of the deformation in any field of gravitational potential V:

$$\iiint V dv \quad \text{(taken throughout the deformation)}$$

$$=\iiint VJ\, d\xi\, d\eta\, d\zeta$$

$$=\iint d\eta\, d\zeta \int_{\alpha}^{\alpha+\delta\xi} VJ\, d\xi$$

$$=\iint d\eta\, d\zeta \left\{V(\alpha,\eta,\zeta)\,J(\alpha,\eta,\zeta)\,\delta\xi+\frac{\delta\xi^2}{2!}\frac{\partial}{\partial\alpha}(VJ)+\ldots\right\}.$$

Hence, if the limits of η and ζ do not involve α (and we suppose the coordinates chosen so that this is so), the energy of the deformation

$$=\text{energy considered as surface density}$$

$$+\frac{1}{2!}\frac{\partial}{\partial\alpha}\{\text{energy of } \delta\xi^2 \text{ considered as surface density}\}$$

$$+\text{etc.}$$

4. As a fuller illustration of the method, I shall now find the relation between $\omega^2/2\pi$ and the ellipticity in the case of Maclaurin's ellipsoid.

The oblate ellipsoid of revolution may be written

$$r^2=\frac{a^2}{1+\rho\cos^2\vartheta}.$$

Then $$\frac{r}{a}=1-\frac{\rho}{2}\mu^2+\frac{3}{8}\rho^2\mu^4+\frac{5}{16}\rho^3\mu^6+\frac{35}{128}\rho^4\mu^8.$$

Now $\mu^2 = \frac{2}{3} P_2 + \frac{1}{3}$,

$$\mu^4 = \frac{8}{35} P_4 + \frac{4}{7} P_2 + \frac{1}{5},$$

$$\mu^6 = \frac{16}{231} P_6 + \frac{24}{77} P_4 + \frac{10}{21} P_2 + \frac{1}{7},$$

$$\mu^8 = \frac{128}{6435} P_8 + \frac{64}{495} P_6 + \frac{48}{143} P_4 + \frac{40}{99} P_2 + \frac{1}{9}.$$

If the equation be written $r = a + \delta r$ we get

$$\frac{\delta r}{a} = \left(-\frac{\rho}{6} + \frac{3}{40}\rho^2 - \frac{5}{112}\rho^3 + \frac{35}{1152}\rho^4\right)$$
$$+ \left(-\frac{\rho}{3} + \frac{3}{14}\rho^2 - \frac{50}{336}\rho^3 + \frac{175}{1584}\rho^4\right) P_2$$
$$+ \left(\frac{3}{35}\rho^2 - \frac{15}{154}\rho^3 + \frac{210}{2288}\rho^4\right) P_4$$
$$+ \left(-\frac{5}{231}\rho^3 + \frac{35}{990}\rho^4\right) P_6 + \frac{7}{1287}\rho^4 P_8,$$

$$\left(\frac{\delta r}{a}\right)^2 = \left(\frac{\rho^2}{20} - \frac{3}{56}\rho^3 + \frac{29}{576}\rho^4\right) + P_2\left(\frac{\rho^2}{7} - \frac{5}{28}\rho^3 + \frac{145}{792}\rho^4\right)$$
$$+ P_4\left(\frac{2}{35}\rho^2 - \frac{9}{77}\rho^3 + \frac{87}{572}\rho^4\right)$$
$$+ P_6\left(-\frac{6}{231}\rho^3 + \frac{29}{495}\rho^4\right) + P_8 \frac{58}{6435}\rho^4,$$

$$\left(\frac{\delta r}{a}\right)^3 = \left(-\frac{\rho^3}{56} + \frac{\rho^4}{32}\right) + P_2\left(-\frac{5}{84}\rho^3 + \frac{5}{44}\rho^4\right)$$
$$+ P_4\left(-\frac{3}{77}\rho^3 + \frac{27}{286}\rho^4\right)$$
$$+ P_6\left(-\frac{2}{231}\rho^3 + \frac{18}{495}\rho^4\right) + P_8 \frac{4\rho^4}{715},$$

$$\left(\frac{\delta r}{a}\right)^4 = \frac{\rho^4}{144} + P_2\frac{5\rho^4}{198} + P_4\frac{3\rho^4}{143} + P_6\frac{4\rho^4}{495} + P_8\frac{8\rho^4}{6435}.$$

Let us write

$$\delta r = a_0 + a_2P_2 + a_4P_4 + a_6P_6 + a_8P_8,$$
$$(\delta r)^2 = b_0 + b_2P_2 + \ldots\ldots\ldots\ldots + b_8P_8,$$
$$(\delta r)^3 = c_0 + \ldots\ldots\ldots\ldots\ldots\ldots + c_8P_8,$$
$$(\delta r)^4 = d_0 + \ldots\ldots\ldots\ldots\ldots\ldots + d_8P_8.$$

The potential at internal points of the deformation δr reated as surface density is (density unity)

$$4\pi\left\{\frac{a_2r^2P_2}{5a} + \frac{a_4r^4P_4}{9a^3} + \frac{a_6r^6P_6}{13a^5} + \frac{a_8r^8P_8}{17a^7}\right\}.$$

Hence the potential of the deformation at inside points is

$$4\pi\left|\begin{array}{l} \frac{r^2P_2}{5}\left(\frac{a_2}{a} - \frac{b_2}{2a^2} + \frac{c_2}{3a^3} - \frac{d_2}{4a^4}\right) \\ + \frac{r^4P_4}{9}\left(\frac{a_4}{a^3} - \frac{3b_4}{2a^4} + \frac{4c_4}{2a^5} - \frac{5d_4}{2a^6}\right) \\ + \frac{r^6P_6}{13}\left(\frac{a_6}{a^5} - \frac{5b_6}{2a^6} + \frac{5c_6}{a^7} - \frac{35}{4}\frac{d_6}{a^8}\right). \\ + \frac{r^8P_8}{17}\left(\frac{a_8}{a^7} - \frac{7b_8}{2a^8} + \frac{28}{3}\frac{c_8}{a^9} - 21\frac{d_9}{a^{10}}\right) \end{array}\right|$$

(correct to the fourth order of ρ)

$$= (\text{say})\ 4\pi\left(A_2r^2P_2 + A_4r^4P_4 + A_6r^6P_6 + A_8r^8P_8\right)$$
$$= 4\pi A_2r^2P_2,$$

all the rest vanishing identically.

If we add the potential of the sphere $r = a$ and also the centrifugal potential we get, as the equation for ω,

$$\frac{\omega^2 - 2\pi}{3}r^2 + \left(4\pi A_2 - \frac{\omega^2}{3}\right)r^2P_2 = \text{constant } C.\ldots\ldots(1)$$

on the surface $r = a + \delta r$.

Taking $\mu = \frac{1}{2}\pi$ we find for C the value

$$\frac{\omega^2 - 2\pi}{3} + \tfrac{1}{2}\left(\frac{\omega^2}{3} - 4\pi A_2\right).$$

Multiply both sides of (1) by $(1+\rho\mu^2)$. Then

$$C(1+\rho\mu^2) = \frac{\omega^2 - 2\pi}{3} + \left(4\pi A_2 - \frac{\omega^2}{3}\right)\left(\tfrac{3}{2}\mu^2 - \tfrac{1}{2}\right).$$

Equating to zero the coefficient of μ^2 we get

$$C\rho = \tfrac{3}{2}\left(4\pi A_2 - \frac{\omega^2}{3}\right),$$

a being now put equal to unity; whence

$$\frac{\omega^2}{4\pi}(\rho + 1) = A_2(\rho + 3) + \frac{\rho}{3}.$$

Now
$$A_2 = -\frac{\rho}{15} + \frac{\rho^2}{35} - \frac{\rho^3}{63} + \frac{\rho^4}{99}.$$

Thus
$$\frac{\omega^2}{4\pi} = \frac{\rho}{\rho+1}\left(\frac{2}{15} + \frac{2}{105}\rho - \frac{2}{105}\rho^2 + \frac{10}{33.21}\rho^3\right),$$

which is known to be the correct value to the fourth order. In fact, if we put $\rho + 1 = (1-\rho_0)^{-2}$ so that the surface is written

$$r^2 = \frac{a^2}{\{\cos^2\vartheta/(1-\rho_0)^2\} + \sin^2\vartheta},$$

we obtain

$$\frac{\omega^2}{4\pi} = \rho_0\left\{\frac{4}{15} - \frac{2}{35}\rho_0 - \frac{8}{105}\rho_0^2 - \frac{38}{693}\rho_0^3\right\}$$

$$= \cos j \sum_1^\infty (2n-1)!\sin^{2n} j/(2n+1)(2n+3)\{(n-1)!\}^2 2^{2n-3},$$

as far as the fourth order, where $\cos j = 1 - \rho_0$.

Note 1. The vanishing of the harmonics of higher order than the second is of course the expression of the fact that the potential of an ellipsoid at internal points is a quadratic

function of the coordinates, and might have been expected. It can be proved that the quantities A_4, A_6, etc., vanish to any order by the method of the following note.

Note 2. This method serves to obtain the accurate expression for $\omega^2/2\pi$ in finite terms:

$$A_2 = \tfrac{1}{5}\{a_2 - \tfrac{1}{2}b_2 + \tfrac{1}{3}c_2 - \tfrac{1}{4}d_2 \ldots ad\ inf.\}$$

where

$$\frac{\delta r}{a} = a_0 + a_2 P_2 + a_4 P_4 + \ldots,$$

$$\left(\frac{\delta r}{a}\right)^2 = b_0 + b_2 P_2 + \ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots;$$

whence

$$A_2 = \tfrac{1}{2}\int_{-1}^{+1} P_2 \left\{\frac{\delta r}{a} - \tfrac{1}{2}\left(\frac{\delta r}{a}\right)^2 + \tfrac{1}{3}\left(\frac{\delta r}{a}\right)^3 \ldots\right\} d\mu$$

$$= \tfrac{1}{2}\int_{-1}^{+1} \log\left(1 + \frac{\delta r}{a}\right) P_2\, d\mu$$

$$= -\tfrac{1}{4}\int_{-1}^{+1} \log(1 + \rho\cos^2\vartheta)\, P_2\, d\mu.$$

Expanding the logarithm and integrating term by term, we have

$$A_2 = -\left\{\frac{\rho}{3.5} - \frac{\rho^2}{5.7} + \frac{\rho^3}{7.9} \ldots ad\ inf.\right\}$$

$$= -\frac{\rho}{6} + \frac{\rho+1}{2}\left\{\frac{\tan^{-1}\sqrt{\rho} - \sqrt{\rho} + \frac{1}{3}(\sqrt{\rho^3})}{(\sqrt{\rho})^3}\right\}$$

on decomposing the terms and using Gregory's series. Then

$$\frac{\omega^2}{2\pi} = \frac{2\rho}{3(\rho+1)} + 2A_2\frac{\rho+3}{\rho+1}$$

$$= (3\cot^2\phi + 1)\,\phi\cot\phi - 3\cot^2\phi,$$

if we put $\rho = \tan^2\phi$.

Note 3. It is not difficult to exhibit M. Poincaré's results for the potential and energy of a double layer (pp. 349 and 350 of the memoir already referred to) as an example of this method.

SYLVESTER'S THEOREM RELATING TO BERNOULLIAN NUMBERS.

By *Thorold Gosset.*

Sylvester, in 1861 (*Comptes Rendus*, vol. lii., p. 162), enunciated the following theorem:—If p be a prime and B_n be the n^{th} Bernoullian number, then when p^i is a divisor of n and $(p-1)$ is not a divisor of $2n$, the numerator of B_n is divisible by p^i.

This theorem was proved by Dr. Glaisher, in 1899, in the first of two papers on "Fundamental theorems relating to Bernoullian numbers," which will be found in the *Messenger of Mathematics*, vol. xxix., pp. 47 and 129. In these papers Dr. Glaisher also proves that if p is an odd prime and $j=\frac{1}{2}(p-1)$, then

$$\frac{B_n}{n}=(-1)^{tj}\frac{B_{n-tj}}{n-tj} \pmod{p},$$

t being an integer such that $n-tj$ is positive, and n not being a multiple of j. This result includes Sylvester's theorem as a particular case.

After Dr. Glaisher's complete investigation, it may well seem superfluous to give a second proof of Sylvester's theorem; but as this theorem remained so long unproved and Professor Adams did not succeed in obtaining a proof even for the case where $i=1$ (see *Crelle's Journal*, vol. lxxxv., p. 269), the fact that it admits of such an elementary proof as the following may not be without interest.

Let Σx^n stand for $1^n+2^n+3^n+\ldots+(x-1)^n$: we shall, in the first instance, deduce the value of $\Sigma p^r p^i \pmod{p^{i+1}}$, where r is an integer and p a prime.

1. When r is a multiple of $p-1$. If k is a prime to p, $k^{(p-1)p^i}\equiv 1 \pmod{p^{i+1}}$, accordingly every one of the $(p-1)$ terms in the series reduces to 1 $(\mathrm{mod}\, p^{i+1})$, therefore

$$\Sigma p^r p^i \equiv (p-1) \pmod{p^{i+1}}.$$

2. When r is not a multiple of $(p-1)$. Let k be any primitive root of p, then the first $(p-1)$ natural numbers are (in a different order) congruent to $1, k, k^2, \ldots, k^{p-2} \pmod p$, or to $1, k, a_2p+k^2, a_3p+k^3, \ldots, a_{p-2}p+k^{p-2} \pmod{p^{i+1}}$, $a_2, a_3, \ldots, a_{p-2}$ being integers, therefore

$$\Sigma p^r p^{is} \equiv 1 + k^r p^{is} + \{a_2 p + k^2\}^r p^{2s} + \{a_3 p + k^3\}^r p^{3s} + \ldots + \{a_{p-2} + k^{p-2}\}^r p^{(p-2)s} \pmod{p^{i+1}}$$

$$\equiv 1 + k^r p^{is} + k^{2r} p^{is} + k^{3r} p^{is} + \ldots + k^{(p-2)r} p^{is} \pmod{p^{i+1}},$$

for, if the terms within the brackets be expanded by the binomial theorem, all except those retained in the last expression have the factor p^{i+1}, for $p^n/n!$ is always divisible by p. Hence

$$(k^r p^{is} - 1)\,\Sigma p^r p^{is} \equiv k^{r(p-1)p^{is}} - 1 \equiv 0 \pmod{p^{i+1}}.$$

But, since r is not a multiple of $(p-1)$ and k is a primitive root of p, $(k^r p^{is} - 1)$ is not divisible by p, therefore

$$\Sigma p^r p^{is} \equiv 0 \pmod{p^{i+1}}.$$

As we only require the values of $\Sigma p^r p^{is}$ in the cases where r is even we shall, in the remainder of the proof, write $2s$ for r.

If Σx^{2n} be expressed in descending powers of x, we have

$$\Sigma x^{2n} = \frac{x^{2n+1}}{n+1} - \frac{x^{2n}}{2} + \frac{B_1}{2!}\,2n\,x^{n-1} - \frac{B_2(2n)!}{4!\,(2n-3)!}\,x^{n-3} + \frac{B_3(2n)!}{6!\,(2n-5)!} - \ldots.$$

The last few terms are easily seen to be

$$-\frac{B_{n-3}}{5!}\,2n\,(2n-1)\,(2n-2)\,(2n-3)\,(-1)^n x^5 + \frac{B_{n-1}}{3!}\,2n\,(2n-1)\,(-1)^n x^3 - B_n(-1)^n x.$$

Writing p for x, and sp^i for n, we have

$$(-1)^{sp^i}\Sigma p^{2sp^i} = -B_{sp^i}.p + \frac{B_{sp^i-1}}{3!}2sp^i(2sp^i-1).p^3$$

$$-\frac{B_{sp^i-2}}{5!}2sp^i(2sp^i-1)(2sp^i-2)(2sp^i-3).p^5+\ldots.$$

Now, every term on the right-hand side after the first is divisible by p^{i+1}, for if n is an odd number greater than 1, $p^n/n!$ is always divisible by p^2, even if p be 2 or 3, and *a fortiori* if p be a greater prime; and it follows from Staudt's theorem that no Bernoullian number can contain p more than once in the denominator, therefore

$$p.B_{sp^i} \equiv (-1)^{sp^i+1}\Sigma p^{2sp^i} \pmod{p^{i+1}}.$$

Hence, if $2s$ is a multiple of $p-1$,

$$p.B_{sp^i} \equiv (-1)^{sp^i+1}\{p-1\} \pmod{p^{i+1}}$$

and p divides the denominator of B_{sp^i}.

If $2s$ is not a multiple of $(p-1)$

$$p.B_{sp^i} \equiv 0 \pmod{p^{i+1}},$$

therefore

$$B_{sp^i} \equiv 0 \pmod{p^i}$$

and p^i divides the numerator of B_{sp^i}.

It is perhaps worth noticing that when p is a prime of the form $4n-1$ we can always ascertain the value of $B_n \pmod{p}$, and consequently by Glaisher's theorem of $B_{n+ij} \pmod{p}$, as the number of classes of the negative determinant $-p$ is congruent to $2B_n \pmod{p}$ if p is of the form $8m+7$, and to $-6B_n \pmod{p}$ if p is greater than 3 and of the form $8m+3$. This may be proved as follows.

Since $\frac{1}{2}\{p(p-1)\}+1 = 8n^2-6n+2$ (an even number)

$$(-1)^{4n^2-3n+1}\Sigma p^{\frac{1}{2}\{p(p-1)\}+1} = -B_{4n^2-3n+1}.p$$

$$+\frac{B_{4n^2-3n}}{3!}\left[\tfrac{1}{2}\{p(p-1)\}+1\right]\tfrac{1}{2}\{p(p-1)\}p^3$$

$$-\frac{B_{4n^2-3n-1}}{5!}\left[\tfrac{1}{2}\{p(p-1)\}+1\right]\tfrac{1}{2}\{p(p-1)\}$$

$$\times\left[\tfrac{1}{2}\{p(p-1)\}-1\right]\left[\tfrac{1}{2}\{p(p-1)\}-2\right]p^5+\ldots.$$

Here, as before, every term on the right-hand side except the first is at least divisible by p^2. Consequently

$$\Sigma p^{\frac{1}{2}[p(p-1)]+1} \equiv (-1)^{4n^2-3n} B_{4n^2-3n+1} \cdot p \pmod{p^2}$$

$$\equiv (-1)^n B_{4n^2-3n+1} \cdot p \pmod{p^2}.$$

If k be any number prime to p, $k^{\frac{1}{2}[p(p-1)]} \equiv \pm 1 \pmod{p^2}$ and $k^{\frac{1}{2}[p(p-1)]+1} \equiv \pm k \pmod{p^2}$, the sign in each case being identical with that of $\left(\frac{k}{p}\right)$, consequently

$$\left\{\begin{matrix}\text{sum of quadratic residues} \\ \text{of numbers less than } p\end{matrix}\right\} - \left\{\begin{matrix}\text{sum of non-residues} \\ \text{of numbers less than } p\end{matrix}\right\}$$

$$\equiv \Sigma p^{\frac{1}{2}[p(p-1)]+1} \pmod{p^2}.$$

Now, Kronecker has shown ("Zur Theorie der elliptishen Functionen," *Berlin Sitzungsberichte* for 1885, p. 772) that the value of the quantity on the left-hand side is $-hp$ or $-\frac{1}{3}(hp)$, accordingly as p is of the form $8m+7$ or $8m+3$, where h is the number of classes of the determinant $-p$. Also, if p is of the form $8m+7$ n is even, and if p is of the form $8m+3$ n is odd. Therefore, if p is of the form $8m+7$,

$$-hp \equiv pB_{4n^2-3n+1} \pmod{p^2},$$

and

$$h \equiv -B_{4n^2-3n+1} \pmod{p}.$$

Similarly, if p is of the form $8n+3$,

$$h \equiv 3B_{4n^2-3n+1} \pmod{p}.$$

The last congruence is true even for the case when $p=3$, as can be easily verified.

Since $4n^2-3n+1=(2n-1)^2+n=\frac{1}{2}(p-1)+n$,

we have, from Glaisher's theorem,

$$\frac{(-1)^n B_n}{n} \equiv \frac{(-1)^{4n^2-3n+1} B_{4n^2-3n+1}}{4n^2-3n+1} \pmod{p}$$

except in the case when $p=3$, for in no other case can n be divisible by $2n-1$.

Therefore, with this exception, we have

$$B_{4n^2-3n+1} \equiv -\frac{4n^2-3n+1}{n} B_n \pmod{p},$$

for n and $4n^2-3n+1$ differ by $(2n-1)^2$, an odd number, and $(-1)^n$ and $(-1)^{4n^2-3n+1}$ are therefore of opposite signs.

Also $$\frac{4n^2-3n+1}{n} = \frac{(4n-1)(4n-2)+2}{(4n-1)+1} \equiv 2 \pmod{p},$$

therefore $$B_{4n^2-3n+1} \equiv -2B_n \pmod{p},$$

if p is not equal to 3. Consequently, when p is of the form $8m+7$,

$$h \equiv 2B_n \pmod{p}.$$

If p is of the form $8m+3$ and greater than 3,

$$h \equiv -6B_n \pmod{p}.$$

It need scarcely be remarked that h, being the difference between the number of residues less than $\frac{1}{2}p$ and the number of non-residues less than $\frac{1}{2}p$, is itself always less than $\frac{1}{2}p$ and is consequently uniquely determined by the above congruences.

For example, if $p=47$, then $n=12$, therefore

$$h \equiv 2B_{12} \equiv \frac{236364091}{1365} \equiv \frac{10}{2} \equiv 5 \pmod{47},$$

therefore $$h=5,$$

the classes, in fact, being $(1, 0, 47)$, $(3, \pm 1, 16)$, $(7, \pm 3, 8)$. If $p=19$, then $n=5$, therefore

$$h \equiv -6B_5 \equiv \frac{-5}{11} \equiv \frac{33}{11} \equiv 3 \pmod{19},$$

therefore $$h=3,$$

the classes, in fact, being $(1, 0, 19)$, $(4, \pm 1, 5)$.

THE DEGENERATE APOLAR LOCUS OF TWO APOLAR TRIADS OF POINTS ON A CONIC.

By *William P. Milne, M.A., D.Sc.*, Clifton College.

1. The following are properties of two apolar triads of points on a conic, in addition to those given by Dr. W. Franz Meyer in his treatise: *Apolarität und rationale Curven*, and Rev. C. F. Russell in the *Proceedings of the London Mathematical Society*, ser. 2, vol. iv., p. 342.

2. *Apolar locus of two class-cubics.*

Let Σ_1 and Σ_2 be two class-cubics, then the cubic locus described by P, which moves so that the tangents from P to Σ_1 apolarly separate the tangents from P to Σ_2, is called the apolar locus of the two class-cubics Σ_1 and Σ_2.

3. *Case when Σ_1 and Σ_2 degenerate into two triads of points on a given conic apolar to one another.*

Let Σ_1 degenerate into the three points D, E, F on the given conic.

Let Σ_2 degenerate into the three points L, M, N on the given conic.

Then, since D, E, F is a triad apolar to L, M, N, it is plain that, if P be a point on the conic, the pencil $P[DEF]$ is apolar to the pencil $P[LMN]$. Hence the conic itself will be part of the apolar locus of the two degenerate class-cubics D, E, F and L, M, N. But the apolar locus of two class-cubics is a cubic curve. Consequently all points P not on the conic, such that $P[DEF]$ is apolar to $P[LMN]$, must lie on a straight line.

We shall call this line the apolar line of D, E, F and L, M, N.

4. *The apolar line passes through the respective poles with respect to the conics of the lines joining the Hessian points of the two given triads D, E, F and L, M, N.*

Let A, B be the Hessian points of the triad D, E, F.

Let C be the pole of the line AB with respect to the conic.

Take ABC as triangle of reference.

Let the conic be parametrically represented in the form

$$x = t^2, \quad y = 1, \quad z = 2t \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and let the parameters of D, E, F be given by

$$t^3 + 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

while those of L, M, N are given by

$$t^3+3pt^2+3qt+1=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

it being noted that (2) and (3) are apolar to one another.

Then, by (1) and (2), the pencil $C[DEF]$ is given by the equation

$$x^3-y^3=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Also, by (1) and (3), the pencil $C[LMN]$ is given by the equation

$$x^3+3\,(2q-3p^2)\,x^2y+3\,(3q^2-2p)\,xy^2-y^3=0 \ldots(5).$$

Now (4) and (5) are apolar, hence $C[DEF]$ is apolar to $C[LMN]$ and the theorem is established.

5. *To find where the apolar line cuts the sides of the triangles DEF and LMN.*

Let the apolar line cut EF in P. Then $P[DEF]$ is apolar to $P[LMN]$. Hence PE is the second polar of PD with respect to the pencil $P[LMN]$.

Now it is a well-known result in the theory of class-cubics that if a variable point P moves on a fixed line l, and if for every position of P the second polar PX of l is taken with respect to the pencil of tangents from P to a class-cubic Σ, then PX passes through the pole of l with respect to Σ.

In the above case, therefore, *the apolar line meets EF in the point in which EF is met by the line joining D to the pole of EF with respect to the triangle LMN.*

6. *To find where the apolar line cuts the nine lines joining D, E, F to L, M, N.*

Take, for instance, the line DL.

Let FE meet DL in U.

Let NM meet DL in V.

Let H be the harmonic conjugate of U with respect to E, F.

Let K be the harmonic conjugate of V with respect to M, N.

Let HK cut DL in P.

Then, since PU, PH harmonically separate both PE, PF and PM, PN, therefore

$$P[DEF] \text{ is apolar to } P[LMN].$$

Thus P is a point on the apolar line.

A NOTE ON EQUIPOTENTIAL CURVES.

By *G. N. Watson.*

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1. In the course of a recent investigation, I found that it was necessary to obtain certain results concerning a class of equipotential curves; these results and the methods of obtaining them seemed to be of sufficient interest to justify embodying them in a separate note.

The only paper with which I am acquainted concerning the topography of equipotential curves is by C. E. Brooks;* it does not however deal with curves of the type which will be the subject of our investigation.

The problem which we shall discuss is the following:—

Given the curve whose equation in complex coordinates is

$$\left|\frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}\right| = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where α, β, γ, δ are complex quantities connected by the relation†

$$\alpha\beta = \gamma\delta \quad \ldots\ldots\ldots\ldots\ldots\ldots (1a).$$

Is it possible to start at $z=0$ and travel along the curve (1) to a point at infinity in any direction? And if this be impossible, and we travel from $z=0$ to infinity in such a way that on the path

$$\left|\frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}\right| \leq K,$$

what is the smallest possible value obtainable for K by choosing the path in an appropriate manner?

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* *Proceedings of the American Philosophical Society*, vol. xliii., no. 177 (1904); some diagrams of equipotential curves due to two charges are to be found in most text-books on Electricity.

† This equation determines $\arg\delta$ *uniquely* in terms of $\arg\alpha$, $\arg\beta$, $\arg\gamma$.

[The curve, $\left|\frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}\right| = \text{constant}$, represents an equipotential curve of a system of line charges perpendicular to the plane passing through the points α, β, γ, δ, the charges being $+1$, $+1$, -1, -1 per unit length; and our object is to determine a path from the origin to infinity on which the lowest value of the potential on the path may be as large as possible.]

2. Let us put

$$z = re^{i\theta}, \quad \alpha = Ae^{ia}, \quad \beta = Be^{ib}, \quad \gamma = Ce^{ic}, \quad \delta = De^{id},$$

where, from (1*a*),

$$AB = CD, \quad a + b = c + d \ldots\ldots\ldots\ldots\ldots\ldots (1b).$$

The equation (1) may then be written

$$\begin{aligned} &2r^3[A\cos(\theta-a) + B\cos(\theta-b) - C\cos(\theta-c) - D\cos(\theta-d)] \\ &\quad - r^2[A^2 + B^2 - C^2 - D^2 + 4AB\cos(\theta-a)\cos(\theta-b) \\ &\qquad\qquad - 4CD\cos(\theta-c)\cos(\theta-d)] \\ &\quad + 2rAB[A\cos(\theta-b) + B\cos(\theta-a) \\ &\qquad - C\cos(\theta-d) - D\cos(\theta-c)] = 0. \end{aligned}$$

Putting $r\cos\theta = x$, $r\sin\theta = y$, we see that the curve (1) is a circular cubic, passing through the origin. By using (1*b*) the last equation may be reduced to

$$\begin{aligned} &2r^3[A\cos(\theta-a) + B\cos(\theta-b) - C\cos(\theta-c) - D\cos(\theta-d)] \\ &\quad - r^2[A^2 + B^2 - C^2 - D^2 + 4AB(\cos a\cos b - \cos c\cos d)] \\ &\quad + 2rAB[A\cos(\theta-b) + B\cos(\theta-a) \\ &\qquad - C\cos(\theta-d) - D\cos(\theta-c)] = 0 \ldots\ldots\ldots\ldots\ldots (2). \end{aligned}$$

We may write this in the form

$$(x^2 + y^2)(lx + my + n) + l'x + m'y = 0 \ldots\ldots (2a),$$

where

$$\left.\begin{aligned} \tfrac{1}{2}l &= A\cos a + B\cos b - C\cos c - D\cos d \\ \tfrac{1}{2}m &= A\sin a + B\sin b - C\sin c - D\sin d \\ n &= A^2 + B^2 - C^2 - D^2 + 4AB(\cos a\cos b - \cos c\cos d) \\ \tfrac{1}{2}l' &= AB(A\cos b + B\cos a - C\cos d - D\cos c) \\ \tfrac{1}{2}m' &= AB(A\sin b + B\sin a - C\sin d - D\sin c) \end{aligned}\right\} \ldots (2b).$$

It will be possible to travel from the origin to infinity along the curve (1) if, either (i) the curve be unipartite,* or (ii) the curve be bipartite and the origin not on the finite oval.

[The shape of the curve in the two cases is given roughly by Salmon, *Higher Plane Curves* (1873), p. 167. In the first case the curve is called a *serpentine*; in the second the curve consists of a serpentine branch and a finite oval branch.]

To investigate the conditions (i) and (ii), let us turn the coordinate axes of (2a) by the substitution†

$$x=\frac{lx_1-my_1}{\sqrt{(l^2+m^2)}},\quad y=\frac{mx_1+ly_1}{\sqrt{(l^2+m^2)}}.$$

The equation becomes

$$(x_1^2+y_1^2)(x_1+\lambda)+Lx_1+My_1=0\ldots\ldots\ldots(3),$$

where $\lambda=n(l^2+m^2)^{-\frac{1}{2}}$,

$$L=\frac{ll'+mm'}{l^2+m^2},\qquad M=\frac{lm'-l'm}{l^2+m^2}.$$

The asymptote of (3) is $x_1+\lambda=0$; for a given value of x_1 the two corresponding values of y_1 will be real if

$$4(x_1+\lambda)(x_1^3+\lambda x_1+Lx_1)-M^2\leq 0,$$

i.e., if $$4(x_1^2+\lambda x_1)^2+4L(x_1^2+\lambda x_1)-M^2\leq 0.$$

This inequality is satisfied if $2(x_1^2+\lambda x_1)+L$ lies between

$$-(L^2+M^2)^{\frac{1}{2}}\text{ and }(L^2+M^2)^{\frac{1}{2}},$$

i.e., if $(x_1+\frac{1}{2}\lambda)^2$ lies between

$$\tfrac{1}{4}\lambda^2-\tfrac{1}{2}L-\tfrac{1}{2}(L^2+M^2)^{\frac{1}{2}}\text{ and }\tfrac{1}{4}\lambda^2-\tfrac{1}{2}L+\tfrac{1}{2}(L^2+M^2)^{\frac{1}{2}}.$$

The second of these two expressions is positive; let it be k_2^2; if the first is positive let it be k_1^2 (where $k_2>k_1>0$); the values of y_1 will be real only if

$$-k_2<x_1+\tfrac{1}{2}\lambda<-k_1\text{ or if }k_1<x_1+\tfrac{1}{2}\lambda<k_2.$$

---

* This case includes the case when the curve is unicursal and has a double point; it will appear that the curve cannot have a conjugate point; we notice that if $l=m=0$, $\alpha+\beta=\gamma+\delta$, and (1) is satisfied by every point in the plane.

† The square root of a positive quantity is understood to be the positive square root throughout the note; the only other square roots which will occur are those of α, β, γ, δ.

The curve is then bipartite; the asymptote lies between the lines

$$x_1 = -k_2 - \tfrac{1}{2}\lambda, \quad x_1 = -k_1 - \tfrac{1}{2}\lambda,$$

and the origin lies between the lines

$$x_1 = k_1 - \tfrac{1}{2}\lambda, \quad x_1 = k_2 - \tfrac{1}{2}\lambda$$

(for it is readily verified that $k_1 < \frac{1}{2}\lambda < k_2$). It follows at once that it is impossible that (ii) should be the case. We have then to consider the circumstances in which (i) is the case; if

$$\tfrac{1}{4}\lambda^2 - \tfrac{1}{2}L - \tfrac{1}{2}(L^2 + M^2)^{\frac{1}{2}} \leq 0,$$

the values of y_1 will be real if, and only if,

$$-k_2 \leq x_1 + \tfrac{1}{2}\lambda \leq k_2.$$

It is then obvious that the curve cannot have an isolated oval or a conjugate point; that is to say the curve certainly is unipartite if

$$\tfrac{1}{4}\lambda^2 - \tfrac{1}{2}L \leq \tfrac{1}{2}(L^2 + M^2)^{\frac{1}{2}}.$$

We have to express this inequality in terms of α, β, γ, δ; the inequality is equivalent to

$$\tfrac{1}{2}n^2 - (ll' + mm') < \{(l^2 + m^2)(l'^2 + m'^2)\}^{\frac{1}{2}}.$$

Since

$$\tfrac{1}{2}(l \pm im) = Ae^{\pm ia} + Be^{\pm ib} - Ce^{\pm ic} - De^{\pm id},$$

$$\tfrac{1}{2}(l' \mp im') = e^{\mp i(a+b)} AB\left[Ae^{\pm ia} + Be^{\pm ib} - Ce^{\pm ic} - De^{\pm id}\right].$$

we have $$l \mp im' = e^{\mp i(a+b)} AB(l \pm im).$$

The inequality is therefore

$$\tfrac{1}{2}n^2 \leq ll' + mm' + AB(l^2 + m^2).$$

Whence

$$n^2 \leq (l + im)(l' - im') + (l - im)(l' + im') + 2AB(l^2 + m^2),$$

i.e., $$n^2 \leq AB\left[(l + im)^2 e^{-i(a+b)} + (l - im)^2 e^{i(a+b)} + 2(l^2 + m^2)\right],$$

i.e., $$n^2 \leq AB\left[(l + im)\, e^{-\frac{1}{2}i(a+b)} + (l - im)\, e^{\frac{1}{2}i(a+b)}\right]^2,$$

i.e., $$n^2 \leq 16AB\left[(A + B)\cos\tfrac{1}{2}(a - b) - (C + D)\cos\tfrac{1}{2}(c - d)\right]^2.$$

That is to say

$$[n-4A^{\frac{1}{2}}B^{\frac{1}{2}}\{(A+B)\cos\tfrac{1}{2}(a-b)-(C+D)\cos\tfrac{1}{2}(c-d)\}]$$
$$\times[n+4A^{\frac{1}{2}}B^{\frac{1}{2}}\{(A+B)\cos\tfrac{1}{2}(a-b)-(C+D)\cos\tfrac{1}{2}(c-d)\}]\leq 0.$$

Substituting for n, from (2b), on writing $2(\cos a\cos b-\cos c\cos d)$ in the form $\cos(a-b)-\cos(c-d)$, we find that the inequality is

$$[A^2+B^2+2AB\cos(a-b)-4A^{\frac{1}{2}}B^{\frac{1}{2}}(A+B)\cos\tfrac{1}{2}(a-b)$$
$$-C^2-D^2-2CD\cos(c-d)+4C^{\frac{1}{2}}D^{\frac{1}{2}}(C+D)\cos\tfrac{1}{2}(c-d)]$$
$$\times[A^2+B^2+2AB\cos(a-b)+4A^{\frac{1}{2}}B^{\frac{1}{2}}(A+B)\cos\tfrac{1}{2}(a-b)$$
$$-C^2-D^2-2CD\cos(c-d)-4C^{\frac{1}{2}}D^{\frac{1}{2}}(C+D)\cos\tfrac{1}{2}(c-d)]\leq 0,$$

i.e.,

$$[\{A+B-2A^{\frac{1}{2}}B^{\frac{1}{2}}\cos\tfrac{1}{2}(a-b)\}^2$$
$$-\{C+D-2C^{\frac{1}{2}}D^{\frac{1}{2}}\cos\tfrac{1}{2}(c-d)\}^2]$$
$$\times[\{A+B+2A^{\frac{1}{2}}B^{\frac{1}{2}}\cos\tfrac{1}{2}(a-b)\}^2$$
$$-\{C+D+2C^{\frac{1}{2}}D^{\frac{1}{2}}\cos\tfrac{1}{2}(c-d)\}]^2\leq 0.$$

Since

$$A+B\geq\pm 2A^{\frac{1}{2}}B^{\frac{1}{2}}\cos\tfrac{1}{2}(a-b),\quad C+D\geq\pm 2C^{\frac{1}{2}}D^{\frac{1}{2}}\cos\tfrac{1}{2}(c-d),$$

on rejecting positive factors, the inequality becomes

$$[\{A+B-2A^{\frac{1}{2}}B^{\frac{1}{2}}\cos\tfrac{1}{2}(a-b)\}$$
$$-\{C+D-2C^{\frac{1}{2}}D^{\frac{1}{2}}\cos\tfrac{1}{2}(c-d)\}]$$
$$\times[\{A+B+2A^{\frac{1}{2}}B^{\frac{1}{2}}\cos\tfrac{1}{2}(c-d)\}$$
$$-\{C+D+2C^{\frac{1}{2}}D^{\frac{1}{2}}\cos\tfrac{1}{2}(c-d)\}]\leq 0.$$

This may be written

$$|A+B-C-D|\leq 2A^{\frac{1}{2}}B^{\frac{1}{2}}|\cos\tfrac{1}{2}(a-b)-\cos\tfrac{1}{2}(c-d)|\ldots(4).$$

We may also write it in the form

$$[|\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}|^2-|\gamma^{\frac{1}{2}}+\delta^{\frac{1}{2}}|^2][|\alpha^{\frac{1}{2}}-\beta^{\frac{1}{2}}|^2-|\gamma^{\frac{1}{2}}-\delta^{\frac{1}{2}}|^2]\leq 0,$$

i.e., $[|\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}|-|\gamma^{\frac{1}{2}}+\delta^{\frac{1}{2}}|][|\alpha^{\frac{1}{2}}-\beta^{\frac{1}{2}}|-|\gamma^{\frac{1}{2}}-\delta^{\frac{1}{2}}|]\leq 0\ldots(4a).$

If the inequality (4), or its equivalent (4a), is satisfied, we can answer the first question [viz., is it possible to travel along the curve (1) from $z=0$ to $z=\infty$?] in the affirmative; if the inequality is not satisfied the curve is bipartite, and the answer is in the negative.

3. We now have to investigate the second question. To do this we consider the family of curves

$$\left|\frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}\right| = t \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

obtained by varying t; each member of the family except that for which $t=1$ is a bicircular quartic; the curve for which $t=1$ is a bipartite circular cubic; we shall call its serpentine branch S and its closed oval V.

We first determine which members of the family (5) have double points; taking the logarithm of both sides of (5), putting $z=x+iy$, and differentiating partially with regard to x and y, we get two equations to determine the coordinates of the double points; these equations may be written

$$\frac{1}{z-\alpha}+\frac{1}{z-\beta}=\frac{1}{z-\gamma}+\frac{1}{z-\delta},$$

$$\frac{1}{z_1-\alpha_1}+\frac{1}{z_1-\beta_1}=\frac{1}{z_1-\gamma_1}+\frac{1}{z_1-\delta_1},$$

where z, z_1; α, α_1; etc. are conjugate imaginaries; the solutions of these equations are

$$z=\pm\sqrt{(\alpha\beta)}, \qquad z_1=\pm\sqrt{(\alpha_1\beta_1)}.$$

We deduce that the curves which have double points are those for which

$$t=\left|\frac{\alpha^{\frac{1}{2}}-\beta^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}-\delta^{\frac{1}{2}}}\right|^2, \qquad \left|\frac{\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}+\delta^{\frac{1}{2}}}\right|^2 \quad \ldots\ldots\ldots\ldots (6).$$

And, because (4*a*) is *not* satisfied, these values of t are both greater than, or both less than, unity.

Let us suppose that at a point T_1, just outside the oval V, the quantity $\left|\frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}\right|$ is *greater** than unity; we shall show that at all points outside V, which are on the same side of S as V is, t is *greater* than unity; for, if T_2 be such a point, we can join T_1 to T_2 by a curve which does not meet S or V or pass through $\alpha, \beta, \gamma, \delta$. On T_1T_2, t is continuous and not

* Throughout this section we put "greater" in italics when it has to be changed to "less" if this fraction is less than unity at T_1.

equal to 1; therefore t cannot be less than 1 at one point of T_1T_2 and greater than 1 at another point. Now consider the shape of

$$\left|\frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}\right| = t$$

when t is *greater* than 1 and $t-1$ is very small; it is easily seen that the curve must consist of a small oval just outside V and a very large closed branch (on the same side of S as V) which contains the smaller ovals; these two branches are the whole of the curve; for the curve is a quartic, and through *any* point of the plane we can draw a straight line, cutting the small oval in two points and the large branch in two points.

As $|t-1|$ increases the small oval increases and the large branch decreases, and neither the branch nor the oval can have a double point before it happens that, in the course of the increase of $t-1$, the large branch meets the oval (if either the branch or the oval had a double point, we could draw through the double point a straight line which met the quartic in more than four points).

Let us call Δ the double point which occurs when the large branch has diminished and the small branch has increased sufficiently for the two branches to meet.

From (6) the values of t at the double points are

$$t_1, t_2 = \left|\frac{\alpha^{\frac{1}{2}}-\beta^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}-\delta^{\frac{1}{2}}}\right|^2, \qquad \left|\frac{\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}+\delta^{\frac{1}{2}}}\right|^2.$$

We see that, if $\left|\frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}\right| > 1$ just outside V, since Δ is the first double point that occurs as $t-1$ increases, the value of t at Δ is the smaller of the quantities t_1, t_2; if $\left|\frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}\right| < 1$ just outside V, the value of t at Δ is the larger of the quantities t_1, t_2. In either case let the value of t at Δ be t_3.

We now verify that $t_1, t_2 \gtrless 1$ according as

$$\left|\frac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}\right| \gtrless 1$$

just outside V. For $t \gtrless 1$ just outside V according as $t \gtrless 1$ at a point which is at a very great distance from the origin on the same side of S (or the asymptote of S) as V.

In Cartesian coordinates the asymptote of S is

$$lx + my + n = 0.$$

The point with coordinates $(\rho n/l,\ \rho' n/m)$, where ρ, ρ' are large and positive, is on the same side of $lx + my + n = 0$ as the origin; and it is easily shown that at this point $t \gtrless 1$ according as

$$A^2 + B^2 - C^2 - D^2 + 2AB\cos a \cos b - 2CD \cos c \cos d \gtrless 0 \ldots (7).$$

From (6), t_1, $t_2 \gtrless 1$ according as

$$\frac{|\alpha^{\frac{1}{4}} - \beta^{\frac{1}{4}}|^4 + |\alpha^{\frac{1}{4}} + \beta^{\frac{1}{4}}|^4}{|\gamma^{\frac{1}{4}} - \delta^{\frac{1}{4}}|^4 + |\gamma^{\frac{1}{4}} + \delta^{\frac{1}{4}}|^4} \gtrless 1;$$

and this inequality is easily seen to be equivalent to (7).

Now consider what happens as we travel from O (which is on V) to infinity. If

$$|\alpha^{\frac{1}{4}} - \beta^{\frac{1}{4}}| < |\gamma^{\frac{1}{4}} - \delta^{\frac{1}{4}}| \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$t < 1$ at all points outside V which are on the same side of S as V. We can therefore travel from O to a point on S by a path on which $t \leq 1$, and thence to infinity along S.

If

$$|\alpha^{\frac{1}{4}} - \beta^{\frac{1}{4}}| > |\gamma^{\frac{1}{4}} - \delta^{\frac{1}{4}}| \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

$t > 1$ at all points outside V which are on the same side of S as V; we can travel from O to Δ by a path on which $t \leq t_3$, from Δ to a point on S by a path on which $t \leq t_3$, and thence to infinity along S at all points of the path $t \leq t_3$, and we cannot find a value of K less than t_3. Also t_3 is the smaller of the quantities t_1, t_2; and we can show that $t_1 \gtrless t_2$ according as

$$(A + B)\cos\tfrac{1}{2}(c - d) \gtrless (C + D)\cos\tfrac{1}{2}(a - b) \ldots (10).$$

4. Our results may be summarised as follows:

(I.) If either (i) the inequality ($4a$), viz.,

$$[|\alpha^{\frac{1}{4}} + \beta^{\frac{1}{4}}| - |\gamma^{\frac{1}{4}} + \delta^{\frac{1}{4}}|][|\alpha^{\frac{1}{4}} - \beta^{\frac{1}{4}}| - |\gamma^{\frac{1}{4}} - \delta^{\frac{1}{4}}|] \leq 0,$$

is satisfied or (ii) the inequality ($4a$) is not satisfied, but the inequality (8), viz.,

$$|\alpha^{\frac{1}{4}} - \beta^{\frac{1}{4}}| < |\gamma^{\frac{1}{4}} - \delta^{\frac{1}{4}}|$$

is satisfied, it *is* possible to find a path from O to infinity at every point of which $\left|\dfrac{(z-\alpha)(z-\beta)}{(z-\gamma)(z-\delta)}\right| \leq 1$.

(II.) If (4*a*) is not satisfied and if

$$|\alpha^{\frac{1}{4}}-\beta^{\frac{1}{4}}|>|\gamma^{\frac{1}{4}}-\delta^{\frac{1}{4}}|,$$

it is possible to find a path from 0 to infinity at every point of which

$$\left|\frac{(z-\alpha)\,(z-\beta)}{(z-\gamma)\,(z-\delta)}\right|\leq t_3,$$

where t_3 is the smaller of the quantities

$$\left|\frac{\alpha^{\frac{1}{4}}-\beta^{\frac{1}{4}}}{\gamma^{\frac{1}{4}}-\delta^{\frac{1}{4}}}\right|^2, \qquad \left|\frac{\alpha^{\frac{1}{4}}+\beta^{\frac{1}{4}}}{\gamma^{\frac{1}{4}}+\delta^{\frac{1}{4}}}\right|^2.$$

5. The geometrical interpretation of the inequality

$$[\,|\alpha^{\frac{1}{4}}-\beta^{\frac{1}{4}}|-|\gamma^{\frac{1}{4}}-\delta^{\frac{1}{4}}|\,]\,[\,|\alpha^{\frac{1}{4}}+\beta^{\frac{1}{4}}|-|\gamma^{\frac{1}{4}}+\delta^{\frac{1}{4}}|\,]\gtrless 0$$

is worthy of note.

If we call the points α, β, γ, δ in the Argand diagram P, Q, R, S respectively, the condition $\alpha\beta=\gamma\delta$ is the condition that the triangles QOS, POR should be similar;* the inequality is equivalent to

$$[\,|\alpha^{\frac{1}{4}}-\beta^{\frac{1}{4}}|^2-|\gamma^{\frac{1}{4}}-\delta^{\frac{1}{4}}|^2\,]\,[\,|\alpha^{\frac{1}{4}}+\beta^{\frac{1}{4}}|^2-|\gamma^{\frac{1}{4}}+\delta^{\frac{1}{4}}|^2\,]\gtrless 0,$$

i.e.,

$$|\alpha-\beta|^2+|\gamma-\delta|^2-|\alpha^{\frac{1}{4}}\gamma^{\frac{1}{4}}-\beta^{\frac{1}{4}}\delta^{\frac{1}{4}}-\alpha^{\frac{1}{4}}\delta^{\frac{1}{4}}-\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}|^2 - |\alpha^{\frac{1}{4}}\gamma^{\frac{1}{4}}-\beta^{\frac{1}{4}}\delta^{\frac{1}{4}}-\alpha^{\frac{1}{4}}\delta^{\frac{1}{4}}+\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}|^2\gtrless 0.$$

Using the identity $|u+v|^2+|u-v|^2\equiv 2\,\{|u|^2+|v|^2\}$, we get

$$|\alpha-\beta|^2+|\gamma-\delta|^2\gtrless 2\,|\alpha^{\frac{1}{4}}\gamma^{\frac{1}{4}}-\beta^{\frac{1}{4}}\delta^{\frac{1}{4}}|^2+2\,|\alpha^{\frac{1}{4}}\delta^{\frac{1}{4}}-\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}|^2.$$

Now

$$|\alpha^{\frac{1}{4}}\gamma^{\frac{1}{4}}-\beta^{\frac{1}{4}}\delta^{\frac{1}{4}}|^2=|\alpha-\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}\delta^{\frac{1}{4}}\gamma^{-\frac{1}{4}}|\,|\gamma-\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}\delta^{\frac{1}{4}}\alpha^{-\frac{1}{4}}|$$
$$=|\alpha-\delta|.|\gamma-\beta|,$$

since

$$\alpha\beta=\gamma\delta;$$

and similarly

$$|\alpha^{\frac{1}{4}}\delta^{\frac{1}{4}}-\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}|^2=|\alpha-\gamma|.|\beta-\delta|.$$

We may consequently write the inequality in the form

$$PQ^2+RS^2\gtrless 2PS.QR+2PR.QS.$$

Also, assuming that (4*a*) is not satisfied, t_1, t_2 are both less or both greater than unity according as t_1t_2 is greater or less than unity: *i.e.*, t_1, $t_2\gtrless 1$ according as $PQ\gtrless RS$.

There seems to be no convenient way of distinguishing t_1 from t_2, since, by increasing $\arg\alpha$ and $\arg\gamma$ by 2π, we interchange t_1 and t_2.

* Given any four points P, Q, R, S it is possible to find one point O such that the triangles QOS, POR should be similar.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XXXI.

The uniform convergence of Borel's integral.

1. Borel defines the sum of the power series

$$\Sigma a_n x^n \tag{1}$$

as

$$s(x) = \int_0^\infty e^{-t} u(tx)\, dt, \tag{2}$$

where

$$u(x) = \Sigma \frac{a_n x^n}{n!}. \tag{3}$$

In applying this definition we have to distinguish two cases.

(*a*) *The radius of convergence of* (1) *is positive.* In this case (3) is convergent for all values of x.

(*b*) *The radius of convergence of* (1) *is zero.* In this case (3) may converge for all values of x, for some only, or for none. In the first case the original definition applies. In the second case (3) defines a branch of an analytic function, regular about $x=0$, and it may happen that this branch is regular along the line described by tx, when x has some particular value and t varies along the real axis from 0 to ∞. We can then still apply the formula (2), regarding $u(tx)$ as defined by the power series (3) and its analytic continuation along this line. In the third case the sum of (1) can only be defined by some extension of Borel's definition.

2. It has been proved by various writers* that, *if the integral* (2) *is convergent for* $x=1$, *it is uniformly convergent for* $0 < \delta \leq x \leq 1$, however small be δ. It has been stated

* See Phragmen, *Comptes Rendus*, t. 132, p. 1396; Hardy, *Quarterly Journal*, vol. xxxv., p. 44; Bromwich, *Infinite Series*, p. 291 (where the result is made more complete).

that the integral is uniformly convergent for $0 \leq x \leq 1$, and application has been made of theorems in this form.* But the argument used for the interval $\delta \leq x \leq 1$ does not apply to an interval which includes the origin. My object in writing this note is to fill up this gap in the theory.

I observe first that in Case (*a*) the desired extension of the proof presents no difficulty whatever. For, suppose that the radius of convergence of (1) is ρ. Then, if σ is any number greater than $1/\rho$,

$$|a_n| = \epsilon_n \sigma^n,$$

where $\epsilon_n \to 0$ as $n \to \infty$; and from this it follows immediately that

$$e^{-\sigma t x} |u(tx)| \to 0$$

as $t \to \infty$. Thus

$$|u(tx)| < K e^{\sigma t x},$$

where K is a constant. We can choose δ so that

$$\sigma \delta < \tfrac{1}{2};$$

and it follows, by comparison with

$$\int_0^\infty K e^{-\frac{1}{2}t} dt,$$

that (2) is uniformly convergent for $0 \leq x \leq \delta$. As it is known to be uniformly convergent for $\delta \leq x \leq 1$, it is uniformly convergent for $0 \leq x \leq 1$.

An extension of this argument is sufficient to prove that *Borel's integral is uniformly convergent throughout any closed region lying entirely inside the "polygon of summability."*†

3. In Case (*b*), however, when there is no circle of convergence for (1), nor polygon of summability for (3), this argument breaks down. And as the origin is in general a singular point for the function defined by the integral (2), this is not surprising. I shall therefore prove the following theorem.

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* *E.g.*, by the present writer, *Proc. L.M.S.*, vol viii., p. 282.
† Bromwich, *l.c.*, p. 296.

THEOREM. *Let $(u)x$ be a real function of x, continuous for all positive values of x. Then if the integral*

$$\int_0^\infty e^{-t} u(tx)\, dt$$

is convergent for $x=1$, it is uniformly convergent for $0 \leqq x \leqq 1$, that is to say, given ϵ we can find H_ϵ so that

$$\left| \int_{H_1}^{H_2} e^{-t} u(tx)\, dt \right| < \epsilon,$$

for $H_2 \geqq H_1 \geqq H_0$, $0 \leqq x \leqq 1$.

As the integral is known to be uniformly convergent for $\delta \leqq x \leqq 1$ for any positive δ, we may replace 1, in the inequalities contained in the enunciation, by δ.

We can choose ξ and K so that

$$|u(x)| < K, \tag{4}$$

for $0 \leqq x \leqq \xi$. We can then choose positive numbers δ, β so that

$$\delta + (1/\beta) < \tfrac{1}{2}, \tag{5}$$

$$\beta x \log(1/x) < \xi, \tag{6}$$

for $0 \leqq x \leqq \delta$. Thus we might take $\beta = 3$, and then choose η so that

$$\delta < \tfrac{1}{6}, \quad 3x \log(1/x) < \xi \quad (0 \leqq x \leqq \delta).$$

Then

$$|u(tx)| < K, \tag{7}$$

for $0 \leqq t \leqq \beta \log(1/x)$, $0 \leqq x \leqq \delta$.

Now

$$\int_{H_1}^{H_2} e^{-t} u(tx)\, dt = \left(\int_{H_1}^{\beta \log(1/x)} + \int_{\beta \log(1/x)}^{H_2} \right) e^{-t} u(tx)\, dt$$

if $H_1 < \beta \log(1/x) < H_2$. If $H_1 > \beta \log(1/x)$ or $H_2 < \beta \log(1/x)$ no such splitting up of the integral is necessary, and the argument is simplified.

In the first place

$$\left| \int_{H_1}^{\beta \log(1/x)} e^{-t} u(tx)\, dt \right| < K \int_{H_1}^{\infty} e^{-t}\, dt = K e^{-H_1}. \tag{8}$$

In the second place

$$\int_{\beta\log(1/x)}^{H_2} e^{-t}\,u(tx)\,dt = \frac{1}{x}\int_{\beta x\log(1/x)}^{H_2 x} e^{-w/x}\,u(w)\,dw$$

Writing $$e^{w-(w/x)}\,e^{-w}$$

for $e^{-w/x}$ and applying the second Mean Value Theorem, we obtain

$$x^{\beta-1}\,e^{\beta x\log(1/x)}\int_{\beta x\log(1/x)}^{W} e^{-w}\,u(w)\,dw,$$

where $W > \beta x\log(1/x)$. Since

$$x^{\beta-1} = e^{-(\beta-1)\log(1/x)} < e^{-(\beta-1)H_1/\beta},$$

this is less than

$$M\epsilon^{\xi-(\beta-1)H_1/\beta}, \tag{9}$$

where M is an upper limit for

$$\int_{\tau_1}^{\tau_2} e^{-w}\,u(w)\,dw.$$

Thus, from (8) and (9), we obtain

$$\left|\int_{H_1}^{H_2} e^{-t}\,u(tx)\,dt\right| < Ke^{-H_1} + Me^{\xi-(\beta-1)H_1/\beta}. \tag{10}$$

So far we have supposed $H_1 < \beta\log(1/x) < H_2$. If $H_1 \geqq \beta\log(1/x)$, we have

$$\int_{H_1}^{H_2} e^{-t}\,u(tx)\,dt = \frac{1}{x}\int_{H_1 x}^{H_2 x} e^{-w/x}\,u(w)\,dw,$$

which is numerically less than

$$Me^{-H_1+H_1 x}/x < Me^{-H_1+H_1\delta+(H_1/\beta)} < Me^{-\frac{1}{2}H_1}, \tag{11}$$

since $$1/x = e^{\log(1/x)} < e^{H_1/\beta}, \quad \delta + (1/\beta) < \tfrac{1}{2}.$$

Finally, if $H_2 \leq \beta\log(1/x)$, we have

$$\left|\int_{H_1}^{H_2} e^{-t}\,u(tx)\,dt\right| < Ke^{-H_1}. \tag{12}$$

From (10), (11), and (12) it follows that

$$\left| \int_{H_1}^{H_2} e^{-t} u(tx)\, dt \right|$$

is less than one or other of the numbers

$$Ke^{-H_1}, \quad Me^{-\frac{1}{2}H_1}, \quad Ke^{-H_1} + Me^{\xi-(\beta-1)H_1/\beta}.$$

It is evident that, when K, ξ, β, and δ are fixed, we can choose H_0 so that each of these numbers is less than ϵ for $H_2 \geq H_1 \geq H_0$, $0 \leq x \leq \delta$. Thus the theorem is proved.

4. In order to establish the uniform convergence, for $0 \leq x \leq 1$, of Borel's integral (2), we have only to split up $u(tx)$ into its real and imaginary components and apply the theorem to the two integrals thus obtained.

ON THE QUARTIC RESIDUACITY OF $1+i$.

By *Thorold Gosset.*

Let p be a prime of the form $4n+1$ and $r+si$ be any complex number prime to p, then, if $(r+si)^{\frac{1}{4}(p-1)} \equiv x \pmod{p}$ and $p = a^2+b^2$, x must have one of the following sixteen values:—

$$\pm 1, \quad \pm\frac{a}{b}, \quad \pm i, \quad \pm\frac{a}{b}\,i, \quad \frac{\pm a \pm b}{2b}(1 \pm i).$$

By a simple extension of the usual notation, the value of x may be conveniently represented by the symbol $\left(\frac{r+si}{p}\right)_4$.

Since

$$(1+i)^{\frac{1}{4}(p-1)} \equiv \left(\frac{a+b}{a}\right)^{\frac{1}{4}(p-1)} \quad \{\mathrm{mod}\,(a+bi)\},$$

and

$$(1+i)^{\frac{1}{4}(p-1)} \equiv \left(\frac{a-b}{a}\right)^{\frac{1}{4}(p-1)} \quad \{\mathrm{mod}\,(a-bi)\},$$

and $a+b$, $a-b$, and a are composed exclusively of real odd primes, it is possible to apply Gauss' formula to each of the primes of which $a+b$, $a-b$, and a may be composed, and hence to deduce the values of $(1+i)^{\frac{1}{4}(p-1)}$ $\{\mathrm{mod}\,(a+bi)\}$ and $(1+i)^{\frac{1}{4}(p-1)}$ $\{\mathrm{mod}\,(a-bi)\}$ and consequently the values of $\left(\frac{1+i}{p}\right)_4$.

The resulting value of $\left(\frac{1+i}{p}\right)_4$ is independent of the actual factors of $(a+b)$, $(a-b)$, and a, and is given in the following table:

$a+b$	$p=16n+1$	$p=16n+5$	$p=16n+9$	$p=16n+13$
$\pm 1 \pmod{16}$	1	$-\frac{a+b}{2b}(1+i)$	$-\frac{a}{b}i$	$-\frac{a-b}{2b}(1-i)$
$\pm 3 \pmod{16}$	$-\frac{a}{b}$	$\frac{a-b}{2b}(1+i)$	$-i$	$-\frac{a+b}{2b}(1-i)$
$\pm 5 \pmod{16}$	$\frac{a}{b}$	$-\frac{a-b}{2b}(1+i)$	i	$\frac{a+b}{2b}(1-i)$
$\pm 7 \pmod{16}$	-1	$\frac{a+b}{2b}(1+i)$	$\frac{a}{b}i$	$\frac{a-b}{2b}(1-i)$

In the above table a is odd and b even, but no convention is necessary as to the sign of a or b. This applies throughout this paper.

The value of $\left(\frac{1-i}{p}\right)_4$ may be obtained from the above by writing $-i$ for i.

Since $2=(1+i)(1-i)$, we immediately deduce the values of $\left(\frac{2}{p}\right)_4$, namely,

$a+b$	$p=16n+1$	$p=16n+5$	$p=16n+9$	$p=16n+13$
$\pm 1 \pmod 8$	1	$\frac{a}{b}$	-1	$-\frac{a}{b}$
$\pm 3 \pmod 8$	-1	$-\frac{a}{b}$	1	$\frac{a}{b}$

Whence, if $b=8n$, $\left(\frac{2}{p}\right)_4=1$,

„ $b=8n+4$, $\left(\frac{2}{p}\right)_4=-1$,

„ $a+\frac{1}{2}b=4n$, $\left(\frac{2}{p}\right)_4=\frac{a}{b}$,

„ $a+\frac{1}{2}b=4n+2$, $\left(\frac{2}{p}\right)_4=-\frac{a}{b}$.

Again, since $2=-i(1+i)^2$,

$$\left(\frac{2}{p}\right)_8=\left(\frac{-i}{p}\right)_8\left(\frac{1+i}{p}\right)_4, \text{ and as } \left(\frac{-i}{p}\right)_8=1,-i,-1 \text{ or } i,$$

accordingly as $p\equiv 1,\ 9,\ 17,$ or $25 \pmod{32}$, we have the following scheme for the values of $\left(\frac{2}{p}\right)_8$:

$a+b$	$p=32n+1$	$p=32n+9$	$p=32n+17$	$p=32n+25$
$\pm 1 \pmod{16}$	1	$-\frac{a}{b}$	-1	$\frac{a}{b}$
$\pm 3 \pmod{16}$	$-\frac{a}{b}$	-1	$\frac{a}{b}$	1
$\pm 5 \pmod{16}$	$\frac{a}{b}$	1	$-\frac{a}{b}$	-1
$\pm 7 \pmod{16}$	-1	$\frac{a}{b}$	1	$-\frac{a}{b}$

Whence,

if $b=8n$ and $a+\frac{1}{2}b=8m\pm 1$, $\left(\frac{2}{p}\right)_8=1$,

„ $b=8n$ „ $a+\frac{1}{2}b=8m\pm 3$, $\left(\frac{2}{p}\right)_8=-1$,

„ $b=8n+4$ „ $a+\frac{1}{2}b=8m\pm 3$, $\left(\frac{2}{p}\right)_8=\frac{a}{b}$,

„ $b=8n+4$ „ $a+\frac{1}{2}b=8m\pm 3$, $\left(\frac{2}{p}\right)_8=-\frac{a}{b}$.

If p be of the form $8n+5$, $\left(\frac{2}{p}\right)_8$ has no meaning. In this case the values of $\left(\frac{1+i}{p}\right)_4$ will enable us to find the values of $2^{\frac{1}{8}(p+3)} \pmod{p}$; for

$$2^{\frac{1}{8}(p+3)}\equiv(-i)^{\frac{1}{8}(p+3)}(1+i)^{\frac{1}{4}(p+3)}$$

$$\equiv(-i)^{\frac{1}{8}(p+3)}(1+i)\left(\frac{1+i}{p}\right)_4 \pmod{p},$$

and accordingly we obtain this scheme for the values of $2^{\frac{1}{8}(p+3)} \pmod p$:

$a+b$	$p=32n+5$	$p=32n+13$	$p=32n+21$	$p=32n+29$
$\pm 1 \pmod{16}$	$-\frac{a+b}{b}$	$\frac{a-b}{b}$	$\frac{a+b}{b}$	$-\frac{a-b}{b}$
$\pm 3 \pmod{16}$	$\frac{a-b}{b}$	$\frac{a+b}{b}$	$-\frac{a-b}{b}$	$-\frac{a+b}{b}$
$\pm 5 \pmod{16}$	$-\frac{a-b}{b}$	$-\frac{a+b}{b}$	$\frac{a-b}{b}$	$\frac{a+b}{b}$
$\pm 7 \pmod{16}$	$\frac{a+b}{b}$	$-\frac{a-b}{b}$	$-\frac{a+b}{b}$	$\frac{a-b}{b}$

It will be found that these results can be shortly expressed in the following manner:

if $b=16n\pm 2$ and $a+\frac{1}{2}b=4m$, $\quad 2^{\frac{1}{8}(p+3)} \equiv -\frac{a+b}{b} \pmod p$,

„ $b=16n\pm 2$ „ $a+\frac{1}{2}b=4m+2$, $2^{\frac{1}{8}(p+3)} \equiv \frac{a-b}{b} \pmod p$,

„ $b=16n\pm 6$ „ $a+\frac{1}{2}b=4m$, $\quad 2^{\frac{1}{8}(p+3)} \equiv \frac{a+b}{b} \pmod p$,

„ $b=16n\pm 6$ „ $a+\frac{1}{2}b=4m+2$, $2^{\frac{1}{8}(p+3)} \equiv -\frac{a-b}{b} \pmod p$.

By using the values of $\left(\frac{2}{p}\right)_4$ previously obtained, we can now obtain a simple criterion to show which of the two aurifeuillian factors of $2^{\frac{1}{4}(p-1)}+1$ is divisible by p; thus,

if $b=16n\pm 2$, $2^{\frac{1}{4}(p-1)}+2^{\frac{1}{8}(p+3)}+1$ is divisible by p,

„ $b=16n\pm 6$, $2^{\frac{1}{4}(p-1)}-2^{\frac{1}{8}(p+3)}+1$ is divisible by p.

The two aurifeuillian factors may be distinguished by the fact that one, and one only, is divisible by 5; thus,

if $p=32n+5$ (and $a=16n\pm 1$)

or if $p=32n+29$ (and $a=16n\pm 5$),

$2^{\frac{1}{4}(p-1)}+2^{\frac{1}{8}(p+3)}+1$ is divisible by 5,

if $p=32n+13$ (and $a=16n\pm 3$)

or if $p=32n+21$ (and $a=16n\pm7$),

$$2^{\frac{1}{4}(p-1)}-2^{\frac{1}{8}(p+3)}+1 \text{ is divisible by } 5.$$

Consequently,

if $b=16n\pm2$ and $a=16m\pm1$ or $16m\pm5$

or „ $b=16n\pm6$ „ $a=16m\pm3$ „ $16m\pm7$,

p will divide the same aurifeuillian factors as 5;

if $b=16n\pm2$ and $a=16m\pm3$ or $16m\pm7$

or „ $b=16n\pm6$ „ $a=16m\pm1$ „ $16m\pm5$,

p will divide factors which are not divided by 5.

As every aurifeuillian factor divisible by p contains, algebraically, the lowest aurifeuillian factor so divisible, it follows that the factors that p divides, which are of the form $2^{2n+1}\pm2^{n+1}+1$, are either all divisible by 5 or all prime to 5; consequently all primes of the form $8n+5$ can be divided into two distinct classes, those that divide the same aurifeuillian factor as 5 and those that divide the factor prime to 5. For example,

$$107,367,629=3277^2+9830^2=(205.16-3)^2+(614.16+6)^2$$

belongs to the first class, and

$$1429=23^2+30^2=(16+7)^2+(2.16-2)^2$$

belongs to the second class.

When p is a prime of the form $4n+3$, $\left(\frac{1+i}{p}\right)_4$ signifies the value of $(1+i)^{\frac{1}{4}(p^2-1)} \pmod p$, which must have one of the four values ±1, $\pm i$.

Since

$$(1+i)^{p-1}\equiv(2i)^{\frac{1}{2}(p-1)}\equiv2^{\frac{1}{2}(p-1)}.i^{\frac{1}{2}(p-1)}\equiv-i \pmod p$$

in every case,

$$(1+i)^{\frac{1}{4}(p^2-1)}\equiv(-i)^{\frac{1}{4}(p+1)} \pmod p.$$

Hence $\left(\frac{1+i}{p}\right)_4=-i$, -1, i or 1 according as p is of the form $16n+3$, $16n+7$, $16n+11$, $16n+15$.

POINTS IN THE THEORY OF IGNORATION.

By *R. Hargreaves.*

§ 1. We consider first the question under what circumstances energy and kinetic potential are completely given by the sum and difference of two quantities which may be regarded as kinetic and potential energies in the reduced system. That is the case when the original kinetic energy is the sum of detached quadrics, in which the velocities corresponding to ignorable and non-ignorable coordinates appear separately. It is also the case when the system is reducible to such a form by transformation of the ignorable coordinates, and *this reduction is possible under fully defined conditions.* The first statement will be accepted as obvious, the second requires explanation.

The kinetic energy T_0 of the original system being given by $2T_0 = \Sigma a_{qq'}\dot{x}_q\dot{x}_{q'} + 2\Sigma a_{qq'}\dot{x}_q\dot{x}_{q'}$, we use x_p for a coordinate retained, x_{m+r} for a coordinate ignored, where p ranges from 1 to m and r from 1 to $n-m$; and we write ξ_{m+r} for the constant value of $\dfrac{\partial T_0}{\partial \dot{x}_{m+r}}$. The reduced system has a kinetic potential

$$L = T - K + I \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

involving

$$T_0 = T + K \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

as expression for energy; and T, K, I have the following features:—

(*a*) T is quadratic in $\dot{x}_p$ and is the minimum value of T_0 which can be obtained by attributing arbitrary real values to $\dot{x}_{m+r}$.

(*b*) K is quadratic in ξ_{m+r} and is the reciprocal of the part of T_0 which is quadratic in $\dot{x}_{m+r}$.

(*c*) I is bilinear in $\dot{x}_p$ and ξ_{m+r} and

$$= \sum_{p,r} c_{p,m+r}\dot{x}_p\xi_{m+r}/\Delta \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),^*$$

where

$$\Delta = \begin{vmatrix} a_{m+1,m+1} \cdots \\ \cdots\cdots\cdots \\ \cdots\cdots a_{n,n} \end{vmatrix}, \qquad c_{p,m+r} = \begin{vmatrix} a_{m+1,m+1} \cdots \\ \cdots\cdots\cdots \\ a_{p,m+1} \cdots \\ \cdots\cdots a_{n,n} \end{vmatrix} \quad \ldots\ldots (4).$$

that is, the row in $c_{p,m+r}$ beginning with $a_{p,m+1}$ replaces a row beginning with $a_{m+r,m+1}$ in Δ.

* cf. *Phil. Mag.*, April, 1910, p. 486, where also the coefficients of T are given.

T and K evidently have the forms characteristic of kinetic and potential energies, and they completely determine the reduced system not only when I vanishes, as is the case when $a_{p,m+r}=0$ (the case of detached quadrics in T_0), but also when I has the form $\dfrac{df}{dt}$, which yields no force and is of no account as a part of kinetic potential. In view of the arbitrary values of constants ξ_{m+r}, this is true when the coefficient of $\dot{\xi}_{m+r}$ in I has the form $\dfrac{d}{dt}f_{m+r}(x_1 \ldots x_m)$, that is, when

$$\sum_p c_{p,m+r}\dot{x}_p/\Delta = \frac{df_{m+r}}{dt}, \quad \text{or} \quad \sum_p c_{p,m+r}dx_p/\Delta = df_{m+r} \ldots (5).$$

When the necessary conditions are satisfied we may ignore I in (1) and write

$$L = T - K \ldots\ldots\ldots\ldots\ldots\ldots (1a).$$

The velocity $\dot{x}_{m+r}$ is connected with (1) by the form $\dot{x}_{m+r} = \dfrac{\partial K}{\partial \xi_{m+r}} - \dfrac{\partial I}{\partial \xi_{m+r}}$, while (1$a$) implies for the velocity $\dot{y}_{m+r}$ corresponding to $\dot{\xi}_{m+r}$, $\dot{y}_{m+r} = \dfrac{\partial K}{\partial \xi_{m+r}}$; hence

$$\dot{y}_{m+r} = \dot{x}_{m+r} + \frac{\partial I}{\partial \xi_{m+r}} = \dot{x}_{m+r} + \sum_p c_{p,m+r}\dot{x}_p/\Delta = \dot{x}_{m+r} + \frac{df_{m+r}}{dt}.$$

Thus, under the conditions (5), if x_{m+r} is a coordinate so also is y_{m+r}, and it is ignorable. When K is expressed in terms of $\dot{y}_{m+r}$, the part of T_0 which is quadratic in $\dot{x}_{m+r}$ is reproduced with $\dot{y}_{m+r}$ for $\dot{x}_{m+r}$ in virtue of (b). The transformation from x_{m+r} to y_{m+r} is therefore one which expresses T_0 as the sum of detached quadrics in $\dot{x}_p$ and $\dot{y}_{m+r}$, as stated above. Also it is possible under the perfectly definite conditions (5), which make I nugatory as part of a kinetic potential, and this feature may be regarded as pointing to the transformation.

It should be noted as a particular case that when all coordinates but one are ignorable the conditions (5) are necessarily satisfied and I always nugatory.

We may put the matter rather differently, thus:—The minimum T of T_0 is given by $n-m$ equations of type

$$\sum_p a_{p,m+r}\dot{x}_p + \sum_s a_{m+r,m+s}\dot{x}_{m+s} = 0,$$

equivalent on solution to

$$\dot{x}_{m+r} + \sum_p c_{p,m+r}\dot{x}_p/\Delta = 0,$$

with the value of $c_{p,m+r}$ in (4). If we write

$$\dot{x}_{m+r} + \sum_p c_{p,m+r}\dot{x}_p/\Delta = \dot{y}_{m+r},$$

then T_0 assumes the form described above. This transformation of T_0 can always be realised when $\dot{y}$ is treated as an independent symbol, but y will be a coordinate only under the conditions (5). [The first method shows more clearly the definiteness of the process and conditions.]

As an example take the ignoration of z in

$$2T_0 = a\dot{x}^2 + b\dot{y}^2 + c\dot{z}^2 + 2f\dot{y}\dot{z} + 2g\dot{z}\dot{x} + 2h\dot{x}\dot{y},$$

the result of which is

$$2Lc = (ac - g^2)\,\dot{x}^2 + (bc - f^2)\,\dot{y}^2 + 2\,(ch - fg)\,\dot{x}\dot{y} - \zeta^2 + 2\zeta\,(g\dot{x} + f\dot{y}).$$

Here I will be nugatory under the condition $\dfrac{\partial}{\partial y}\dfrac{g}{c} = \dfrac{\partial}{\partial x}\dfrac{f}{c}$, which in fact permits the transformation

$$dZ = dz + \frac{1}{c}\,(g\,dx + f\,dy),$$

and this makes

$$2T_0 = \frac{1}{c}\,\{(ac - g^2)\,\dot{x}^2 + (bc - f^2)\,\dot{y}^2 + 2\,(ch - fg)\,\dot{x}\dot{y}\} + c\dot{Z}^2.$$

The ignoration of y and z in the same form of T_0 yields, for the effective kinetic potential,

$$2L\,(bc - f^2) = (abc + 2fgh - af^2 - bg^2 - ch^2)\,\dot{x}^2 - (c\eta^2 + b\zeta^2 - 2f\eta\zeta),$$

that of y in $2T_0 = a\dot{x}^2 + b\dot{y}^2 + 2c\dot{x}\dot{y}$ yields

$$2Lb = (ab - c^2)\,\dot{x}^2 - \eta^2;$$

in each of these cases I is unconditionally nugatory and here omitted.

§ 2. Another point touches the use of (1) and (2) as a standard of reference for testing the conformity of results yielded by a physical problem, with a dynamical system reduced by ignoration. The typical example is Kelvin's hydrodynamical problem, in which the constants of circulation are taken to represent momenta corresponding to ignored coordinates. Immediate comparison with (1) and (2) involves simply a verification of the positive character of T and K, and of the agreement of coefficients in ξ_p and $-\dot{x}_{m+r}$ in respect to the terms derived from I, which may be written as

$$\sum_{p,r} \gamma_{p,m+r}\dot{x}_p\xi_{m+r},$$

through the formulæ

$$\xi_p = \frac{\partial T}{\partial \dot{x}_p} + \frac{\partial I}{\partial \dot{x}_p}, \qquad \dot{x}_{m+r} = \frac{\partial K}{\partial \xi_{m+r}} - \frac{\partial I}{\partial \xi_{m+r}}.$$

Comparing this immediate reference to the form $T - K + I$ with the indirect and troublesome reference to the presumed original T_0, we note that the number of coefficients in T_0 is the same as in T, K, I, and they are linearly connected. Those of T_0 are subject to conditions needed to secure its positive character, and to no other conditions; and these are exactly represented by the conditions needed to make T and K positive. We draw the conclusion that the coefficients of I are subject to no conditions of magnitude or sign. In conformity with this view we find that the coefficients of I, as expressed in terms of those of T_0, involve ratios only, and ratios which, for any one scheme of ignoration, may have arbitrary values consistently with T_0 being positive; and the various coefficients in I under different circumstances of ignoration appear to constitute the whole group of such arbitrary ratios.

§ 3. It is clear from the argument in § 1 that a transformation of type $x_{m+r} = y_{m+r} + f_{m+r}(x_1 \ldots x_m)$, which replaces ignorable coordinates x_{m+r} by coordinates y_{m+r} also ignorable, leaves T and K unaltered, but increases I by terms $\sum_r \xi_{m+r} \dot{f}_{m+r}$ of no dynamical account. This consideration is of service in the treatment of the system

$$\begin{aligned} 2T_0 = a' \{\Sigma u_1^2 - (\Sigma l u_1)^2\} + b' \{\Sigma u_2^2 - (\Sigma l u_2)^2\} \\ + 2c' (\Sigma u_1 u_2 - \Sigma l u_1 \Sigma l u_2) \\ + a (\Sigma l u_1)^2 + b (\Sigma l u_2)^2 + 2c \Sigma l u_1 \Sigma l u_2 \ldots\ldots (6). \end{aligned}$$

belonging to the problem of two spheres in infinite liquid.* Here $a \ldots c'$ are functions of r, the mutual distance of centres, l, m, r direction-cosines of that distance, and the kinetic potential of relative motion is required. For the purpose of transformation we write $x_2 - x_1 = x \ldots$ and treat $x_1 y_1 z_1$ as ignorable coordinates. Then, with

$$D = a + b + 2c, \quad D' = a' + b' + 2c', \text{ and } \Delta = ab - c^2, \quad \Delta' = a'b' - c'^2 \ldots (7)$$

required later, $2T_0$ is transformed to

$$\begin{aligned} 2T_0' = D' \{\Sigma u_1^2 - (\Sigma l u_1)^2\} + D (\Sigma l u_1)^2 + 2 (b' + c') \Sigma \dot{x} u_1 \\ + 2 (b + c - b' - c') \dot{r} \Sigma l u_1 + b' (\Sigma \dot{x}^2 - \dot{r}^2) + b \dot{r}^2. \end{aligned}$$

* cf. *Quarterly Journal*, vol. xli. (1910), pp. 308–324. The notation is altered from $A \ldots C'$ to $a \ldots c'$, with one exception, viz. c for $-C$ to give symmetry. The reader will see that the present treatment throws some light on the dynamics and ensibly relieves the algebraical work.

L or $T_0' - \Sigma u_1 \dfrac{\partial T_0'}{\partial u_1}$ is then found in terms of $\dot{x}, \dot{y}, \dot{z}$ by using

$$\left.\begin{aligned} &P = \frac{\partial T_0'}{\partial u_1} = D'(u_1 - l\Sigma l u_1) + Dl\Sigma l u_1 \\ &+ (b'+c')(\dot{x} - l\dot{r}) + (b+c)l\dot{r}, \quad \Sigma lP = D\Sigma l u_1 + (b+c)\dot{r}, \\ &\text{and therefore} \\ &u_1 = \frac{1}{D'}\{P - l\Sigma lP - (b'+c')(\dot{x} - l\dot{r})\} + \frac{l}{D}\{\Sigma lP - (b+c)\dot{r}\}, \\ &\text{and} \\ &D'^2\{\Sigma u_1^2 - (\Sigma l u_1)^2\} = \Sigma P^2 - (\Sigma lP)^2 \\ &\quad + (b'+c')^2(\Sigma \dot{x}^2 - \dot{r}^2) - 2(b'+c')(\Sigma \dot{x}P - \dot{r}\Sigma lP), \end{aligned}\right\} \quad \ldots (8)$$

with the result

$$L = \tfrac{1}{2}\left\{\frac{\Delta'}{D'}(\Sigma \dot{x}^2 - \dot{r}^2) + \frac{\Delta}{D}\dot{r}^2\right\} - \tfrac{1}{2}\left\{\frac{\Sigma P^2 - (\Sigma lP)^2}{D'} + \frac{(\Sigma lP)^2}{D}\right\}$$
$$+ \left(\frac{b+c}{D} - \frac{b'+c'}{D'}\right)\dot{r}\Sigma lP + \frac{(b'+c')\,\Sigma \dot{x}P}{D'} \quad \ldots (9a),$$

$$= T - K + \left(\frac{b+c}{D} - \frac{b'+c'}{D'} - r\frac{d}{dr}\frac{b'+c'}{D'}\right)\dot{r}\Sigma lP$$
$$+ \frac{d}{dt}\frac{(b'+c')\,\Sigma xP}{D'} \quad \ldots (9b),$$

$$= T - K + \left(\frac{b+c}{D} - \frac{b'+c'}{D'} - r\frac{d}{dr}\frac{\mu}{\lambda+\mu}\right)\dot{r}\Sigma lP$$
$$+ \left(\frac{b'+c'}{D'} - \frac{\mu}{\lambda+\mu}\right)\Sigma \dot{x}P + \frac{d}{dt}\frac{\mu \Sigma xP}{\lambda+\mu} \quad \ldots (9c).$$

Since (9*a*) corresponds to the use of $X = x_1$ as ignored co-ordinate, (9*b*) with term $\dfrac{df}{dt}$ suppressed will correspond to the use of $X = x_1 + \dfrac{(b'+c')\,x}{D'} = \dfrac{(a'+c')\,x_1 + (b'+c')\,x_2}{D'}$, and (9*c*) to the use of $X = \dfrac{\lambda x_1 + \mu x_2}{\lambda + \mu}$, where $\lambda : \mu$ is an arbitrary function of r. The second and third of the equivalent values of L, those in (9*b*) and (9*c*) with $\dfrac{df}{dt}$ removed, can be got by direct transformation from T_0 with the use of the alternative values given for X, but the work entailed is more troublesome.

The condition $\frac{b+c}{D} = \frac{b'+c'}{D'} + r\frac{d}{dr}\frac{b'+c'}{D'}$ that I should be nugatory is not generally satisfied.

The kinetic potential in (9) has one more coordinate ignorable, an angle representing rotation about the axis of linear momentum. Thus, taking $P=0=Q$ and using polar coordinates in (9*b*) with term $\frac{df}{dt}$ discarded, we have

$$L = \tfrac{1}{2}\left\{\frac{\Delta}{D}\dot{r}^2 + \frac{\Delta' r^2}{D'}(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta)\right\} - \frac{R^2}{2}\left(\frac{\sin^2\theta}{D'} + \frac{\cos^2\theta}{D}\right) + \left(\frac{b+c}{D} - \frac{b'+c'}{D'} - r\frac{d}{dr}\frac{b'+c'}{D'}\right)R\dot{r}\cos\theta,$$

in which ψ is ignorable. Writing

$$\frac{\partial L}{\partial\dot{\psi}} = \frac{\Delta' r^2}{D'}\dot{\psi}\sin^2\theta = N, \text{ a constant},$$

the ignoration of ψ gives

$$\left.\begin{aligned} L' = {} & \tfrac{1}{2}\left\{\frac{\Delta}{D}\dot{r}^2 + \frac{\Delta'}{D'}r^2\dot{\theta}^2\right) \\ & - \tfrac{1}{2}\left\{R^2\left(\frac{\sin^2\theta}{D'} + \frac{\cos^2\theta}{D}\right) + \frac{N^2 D'}{\Delta' r^2\sin^2\theta}\right\} \\ & + \left(\frac{b+c}{D} - \frac{b'+c'}{D'} - r\frac{d}{dr}\frac{b'+c'}{D'}\right)R\dot{r}\cos\theta, \end{aligned}\right\} \quad \ldots(10).$$

and the energy equation is

$$\left.\begin{aligned} T_0 = {} & \tfrac{1}{2}\left(\frac{\Delta}{D}\dot{r}^2 + \frac{\Delta'}{D'}r^2\dot{\theta}^2\right) \\ & + \tfrac{1}{2}\left\{R^2\left(\frac{\sin^2\theta}{D'} + \frac{\cos^2\theta}{D}\right) + \frac{N^2 D'}{\Delta' r^2\sin^2\theta}\right\}. \end{aligned}\right\}$$

This represents the full extent of the reduction furnished by the method of ignoration.

§ 4. But the angular momentum can be dealt with in a different way in which the coordinates X, Y, Z continue to play a part. The system (6) in fact satisfies conditions of type

$$x_1\frac{\partial T_0}{\partial y_1} - y_1\frac{\partial T_0}{\partial x_1} + x_2\frac{\partial T_0}{\partial y_2} - y_2\frac{\partial T_0}{\partial x_2} + u_1\frac{\partial T_0}{\partial v_1} - v_1\frac{\partial T_0}{\partial u_1} + u_2\frac{\partial T_0}{\partial v_2} - v_2\frac{\partial T_0}{\partial u_2} = 0 \quad \ldots\ldots(11a),$$

in virtue of which there are equations of angular momentum

$$x_1\frac{\partial T_0}{\partial v_1}-y_1\frac{\partial T_0}{\partial u_1}+x_2\frac{\partial T_0}{\partial v_2}-y_2\frac{\partial T_0}{\partial u_2}=N, \text{ a constant}.....(11b).$$

These forms are invariant in the transformation to X, x and are represented by

$$x\frac{\partial T_0'}{\partial y}-y\frac{\partial T_0'}{\partial x}+\dot{x}\frac{\partial T_0'}{\partial \dot{y}}-\dot{y}\frac{\partial T_0'}{\partial \dot{x}}+\dot{X}\frac{\partial T_0'}{\partial \dot{Y}}-\dot{Y}\frac{\partial T_0'}{\partial \dot{X}}=0 \qquad .....(12a),$$

and

$$x\frac{\partial T_0'}{\partial \dot{y}}-y\frac{\partial T_0'}{\partial \dot{x}}+X\frac{\partial T_0'}{\partial \dot{Y}}-Y\frac{\partial T_0'}{\partial \dot{X}}=N.....(12b),$$

or

$$x\frac{\partial T_0'}{\partial \dot{y}}-y\frac{\partial T_0'}{\partial \dot{x}}+XQ-YP=N.....(12c),$$

since $\dfrac{\partial T_0'}{\partial \dot{X}}=P$. Working with $X=x_1$ as above, this is

$$N=b'(x\dot{y}-\dot{x}y)+(b'+c')(xv_1-yu_1)+x_1Q-y_1P,$$

which, by (8),

$$=\frac{\Delta'}{D'}(x\dot{y}-\dot{x}y)+\frac{Q}{D'}\{(a'+c')x_1+(b'+c')x_2\}$$

$$-\frac{P}{D'}\{(a'+c')y_1+(b'+c')y_2\}$$

$$=\frac{\Delta'}{D'}(x\dot{y}-\dot{x}y)+XQ-YP.....(13),$$

with the value of X corresponding to (9*b*). In this method of dealing with the angular momentum it is therefore necessary to reinstate the ignored coordinates. In view of (13), the most convenient form is $X=\{(a'+c')x_1+(b'+c')x_2\}/D'$, which has, moreover, the advantage of reducing to $(m_1x_1+m_2x_2)/(m_1+m_2)$ when r is very great; and the precise form of kinetic potential which corresponds to this use is

$$L=\tfrac{1}{2}\left\{\frac{\Delta'}{D'}(\Sigma\dot{x}^2-\dot{r}^2)+\frac{\Delta}{D}\dot{r}^2\right\}-\tfrac{1}{2}\left\{\frac{\Sigma P^2-(\Sigma lP)^2}{D'}+\frac{(\Sigma lP)^2}{D}\right\}$$

$$+\left(\frac{b+c}{D}-\frac{b'+c'}{D'}-r\frac{d}{dr}\frac{b'+c'}{D'}\right)\dot{r}\Sigma lP...(14).$$

A NEW UNISIGNANT.

By *Thomas Muir, LL.D.*

1. DETERMINANTS having nothing but positive terms in the final form of their development have been noted and considered by Sylvester, Borchardt, and others, the interest attaching to them being their apparent divergence from that fundamental characteristic of general determinants which led Cauchy to view the latter as alternative functions. The object of the present paper is not to reconsider any of the known forms of such unisignants, but to draw attention to a form of very special interest which does not seem to have been noticed before.

2. In the case of this new form the determinant is a function of as many variables as there are co-axial minors, one variable being associated with each co-axial minor through forming part of every element of the latter. It thus may also be described as a determinant with polynomial elements such that all the elements of each co-axial minor have a term in common.

When the determinant is of the third order there are seven (*i.e.*, 2^3-1) co-axial minors, namely,

the determinant itself $|1_1 2_2 3_3|$;

the three two-line minors $|2_2 3_3|$, $|1_1 3_3|$, $|1_1 2_2|$;

and the three one-line minors 1_1, 2_2, 3_3;

so that if the variables corresponding to these be

$$a,$$

$$b_1,\ b_2,\ b_3,$$

$$c_1,\ c_2,\ c_3,$$

the determinant is

$$\begin{vmatrix} a+b_2+b_3+c_1 & a+b_3 & a+b_2 \\ a+b_3 & a+b_1+b_3+c_2 & a+b_1 \\ a+b_2 & a+b_1 & a+b_1+b_2+c_3 \end{vmatrix}.$$

It may be suitably denoted by

$$M(a;\ b_1, b_2, b_3;\ c_1, c_2, c_3).$$

Similarly, the determinant of the second order

$$\begin{vmatrix} a+b_1 & a \\ a & a+b_2 \end{vmatrix},$$

which equals $ab_1+ab_2+b_1b_2$, may be denoted by

$$M(a;\ b_1, b_2)$$

or, since the variables turn out to be similarly involved, by

$$M(a, b_1, b_2).$$

3. The determinant is manifestly axisymmetric. Consequently, when it is of the third order, there are 1.2.3 ways of writing it without introducing new elements or departing from axisymmetry, these six ways corresponding to the six different arrangements of the elements of the diagonal. A deduction readily made from this is that

$$\begin{aligned} &M(a;\ b_1, b_2, b_3;\ c_1, c_2, c_3) \\ &= M(a;\ b_1, b_3, b_2;\ c_1, c_3, c_2), \\ &= M(a;\ b_2, b_1, b_3;\ c_2, c_1, c_3), \\ &= M(a;\ b_2, b_3, b_1;\ c_2, c_3, c_1), \\ &= M(a;\ b_3, b_1, b_2;\ c_3, c_1, c_2), \\ &= M(a;\ b_3, b_2, b_1;\ c_3, c_2, c_1); \end{aligned}$$

in other words, *that* a *being left unmoved we may take the* b's *in any order provided we take the* c's *in the same order.* (1.)

4. Again, if on $M(a;\ b_1, b_2, b_3;\ c_1, c_2, c_3)$ we perform the operations

$$\text{row}_2 - \text{row}_1, \qquad \text{row}_3 - \text{row}_1,$$

it becomes

$$\begin{vmatrix} a+b_2+b_3+c_1 & a+b_3 & a+b_2 \\ -b_2-c_1 & b_1+c_2 & b_1-b_2 \\ -b_3-c_1 & b_1-b_3 & b_1+c_3 \end{vmatrix};$$

and if we then, in order to restore the axisymmetry, perform the operations

$$\mathrm{col}_2 - \mathrm{col}_1, \qquad \mathrm{col}_3 - \mathrm{col}_1,$$

there is obtained

$$\begin{vmatrix} a+b_2+b_3+c_1 & -b_2-c_1 & -b_3-c_1 \\ -b_2-c_1 & b_1+c_2+b_2+c_1 & b_1+c_1 \\ -b_3-c_1 & b_1+c_1 & b_1+c_3+b_3+c_1 \end{vmatrix}.$$

Consequently, as the negative signs in this may legitimately be changed into positive, we have proved that the given unisignant is equal to

$$M(c_1;\ b_1, b_3, b_2;\ a, c_2, c_3);$$

in other words, that a and the first c may be interchanged provided we also interchange the last two b's.

Similarly, by performing the operations

$$\mathrm{row}_1 - \mathrm{row}_2, \qquad \mathrm{row}_3 - \mathrm{row}_2,$$

$$\mathrm{col}_1 - \mathrm{col}_2, \qquad \mathrm{col}_3 - \mathrm{col}_2,$$

we obtain another equivalent form, namely,

$$M(c_2;\ b_3, b_2, b_1;\ c_1, a, c_3);$$

and by performing

$$\mathrm{row}_1 - \mathrm{row}_3, \qquad \mathrm{row}_2 - \mathrm{row}_3,$$

$$\mathrm{col}_1 - \mathrm{col}_3, \qquad \mathrm{col}_2 - \mathrm{col}_3,$$

we obtain $M(c_3;\ b_2, b_1, b_3;\ c_1, c_2, a).$

These three results may be combined by saying that *the* a *may be interchanged with any one of the* c*'s provided we also interchange the two* b*'s which differ in suffix from the* c *chosen.* (II.)

5. Since we may use the result (I.) in connection with each of the three forms obtained in § 4, it follows that we can have in all twenty-four different forms, the first of the seven places being filled by any one of the four variables a, c_1, c_2, c_3, the next three places by any permutation of $b_1 b_2 b_3$, and the last three places by the remaining variables in an order which has to be determined. This, in effect, is not different from

saying that the first place and the last three places may be occupied by any of the twenty-four permutations of $a\ c_1\ c_2\ c_3$, and that the remaining places may be occupied by a properly determined permutation of the b's.

6. In view of the numerous exchanges thus shown to be possible among the variables without altering the value of the function, it is not to be wondered at that the terms of the final development are all affected by the same sign. Expanding the function in a series arranged according to products of the c's, we see that it is equal to

$$c_1c_2c_3+c_1c_2(a+b_1+b_2)+c_1c_3(a+b_1+b_3)+c_2c_3(a+b_2+b_3)$$

$$+c_1\begin{vmatrix} a+b_1+b_3 & a+b_1 \\ a+b_1 & a+b_1+b_2 \end{vmatrix}+c_2\begin{vmatrix} a+b_2+b_3 & a+b_2 \\ a+b_2 & a+b_1+b_2 \end{vmatrix}$$

$$+c_3\begin{vmatrix} a+b_2+b_3 & a+b_3 \\ a+b_3 & a+b_1+b_3 \end{vmatrix}$$

$$+\begin{vmatrix} a+b_2+b_3 & a+b_3 & a+b_2 \\ a+b_3 & a+b_1+b_3 & a+b_1 \\ a+b_2 & a+b_1 & a+b_1+b_2 \end{vmatrix},$$

where it should be noticed that the two-line determinants are

$$M(a+b_1;\ b_3,\ b_2),\quad M(a+b_2;\ b_3,\ b_1),\quad M(a+b_3;\ b_2,\ b_1),$$

and that the three-line determinant, being

$$M(a;\ b_1,\ b_2,\ b_3;\ 0,\ 0,\ 0),$$

is, by § 4, equal to

$$M(0;\ b_1,\ b_3,\ b_2;\ a,\ 0,\ 0),$$

i.e.,

$$\begin{vmatrix} b_2+b_3+a & b_2 & b_3 \\ b_2 & b_1+b_2 & b_1 \\ b_3 & b_1 & b_1+b_3 \end{vmatrix},$$

and therefore is equal to

$$aM(b_1;\ b_2,\ b_3)+4b_1b_2b_3.$$

Thus far the development consequently is

$$\begin{aligned}
&c_1c_2c_3+c_1c_2(a+b_1+b_2)+c_1c_3(a+b_1+b_3)+c_2c_3(a+b_2+b_3)\\
&\quad+c_1(ab_2+b_1b_2+ab_3+b_1b_3+b_2b_3)\\
&\quad+c_2(ab_3+b_2b_3+ab_1+b_2b_1+b_2b_1)\\
&\quad+c_3(ab_1+b_3b_1+ab_2+b_3b_2+b_1b_2)\\
&\quad+a(b_1b_2+b_1b_3+b_2b_3)+4b_1b_2b_3.
\end{aligned}$$

Since, however, the a which occurs in every element of the determinant has been shown to be freely interchangeable with any one of the c's, although they, on the other hand, each occur in only one element of the determinant, it is better to write

$$\begin{aligned}
&c_1c_2c_3+c_1c_2a+c_1c_3a+c_2c_3a\\
&\quad+(c_1+c_2)(c_3+a)b_3+(c_1+c_3)(c_2+a)b_2+(c_1+a)(c_2+c_3)b_1\\
&\quad+(c_1+c_2+c_3+a)(b_1b_2+b_1b_3+b_2b_3)\\
&\quad+4b_1b_2b_3,
\end{aligned}\qquad\text{(III.)}$$

which would be still further improved if we agreed to write c_4 in place of a.

7. If we count $4b_1b_2b_3$ as four terms, the number of terms in the development is seen to be 32. It is, of course, now also obtainable by putting each of the variables equal to 1 in the original determinant, the result being

$$\begin{vmatrix} 4 & 2 & 2\\ 2 & 4 & 2\\ 2 & 2 & 4 \end{vmatrix}, \quad i.e., 2^3.4 \begin{vmatrix} 1 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2 \end{vmatrix}, \quad i.e., 32.$$

8. Passing on, now, to the formation and consideration of the corresponding unisignant of the fourth order, we note first that the number of co-axial minors being in that case $1+4+6+4$, *i.e.*, 2^4-1, we may conveniently denote the corresponding variables by

$$\begin{aligned}
&a,\\
&b_1,\ b_2,\ b_3,\ b_4,\\
&c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ c_6,\\
&d_1,\ d_2,\ d_3,\ d_4,
\end{aligned}$$

and therefore the unisignant itself by

$$M(a;\ b_1, b_2, b_3, b_4;\ c_1, c_2, c_3, c_4, c_5, c_6;\ d_1, d_2, d_3, d_4)$$

or $$M(a;\ b_1b_2b_3b_4;\ c_1c_2c_3c_4c_5c_6;\ d_1d_2d_3d_4),$$

this standing for

$$\begin{vmatrix} \left.\begin{matrix} a+b_2+b_3+b_4 \\ +c_1+c_2+c_3+d_1 \end{matrix}\right\} a+b_3+b_4+c_1 & a+b_2+b_4+c_2 & a+b_2+b_3+c_3 \\ a+b_3+b_4+c_1 & \left.\begin{matrix} a+b_1+b_3+b_4 \\ +c_1+c_4+c_5+d_2 \end{matrix}\right\} a+b_1+b_4+c_4 & a+b_1+b_3+c_5 \\ a+b_2+b_4+c_2 & a+b_1+b_4+c_4 & \left.\begin{matrix} a+b_1+b_2+b_4 \\ +c_2+c_4+c_6+d_3 \end{matrix}\right\} a+b_1+b_2+c_6 \\ a+b_2+b_3+c_3 & a+b_1+b_3+c_5 & a+b_1+b_2+c_6 & \left.\begin{matrix} a+b_1+b_2+b_3 \\ +c_3+c_5+c_6+d_4 \end{matrix}\right\} \end{vmatrix}.$$

9. As before we first take note of the various modes of writing this without introducing new elements or departing from axisymmetry. They are in all 1.2.3.4 in number, each mode corresponding to a different arrangement of the diagonal elements and therefore to a different arrangement of the suffixes of the d's in the diagonal.

Consideration of the first six modes gives

$$\begin{aligned} &M(a;\ b_1b_2b_3b_4;\ c_1c_2c_3c_4c_5c_6;\ d_1d_2d_3d_4) \\ &= M(a;\ b_1b_2b_4b_3;\ c_1c_3c_2c_5c_4c_6;\ d_1d_2d_4d_3) \\ &= M(a;\ b_1b_3b_2b_4;\ c_2c_1c_3c_4c_6c_5;\ d_1d_3d_2d_4) \\ &= M(a;\ b_1b_3b_4b_2;\ c_2c_3c_1c_6c_4c_5;\ d_1d_3d_4d_2) \\ &= M(a;\ b_1b_4b_2b_3;\ c_3c_1c_2c_5c_6c_4;\ d_1d_4d_2d_3) \\ &= M(a;\ b_1b_4b_3b_2;\ c_3c_2c_1c_6c_5c_4;\ d_1d_4d_3d_2), \end{aligned}$$

from which we see (1) that the order of the b's follow the order of the d's, (2) that the c's in the first three places are the same permutation of $c_1c_2c_3$ as the last three d's are of $d_2d_3d_4$, (3) that the last three c's are the same permutation of $c_6c_5c_4$ but written in reverse order.

Proceeding to the seventh mode, namely, that in which the d's occur in the order $d_2d_1d_3d_4$, we find that it equals

$$M(a;\ b_2b_1b_3b_4;\ c_1c_4c_5c_2c_3c_6;\ d_2d_1d_3d_4),$$

and from this, without fresh effort, we obtain five others

all having d_2 in the first of the four last places. Similarly we reach

$$M(a;\ b_3b_1b_2b_4;\ c_2c_4c_6c_1c_3c_5;\ d_3d_1d_2d_4),$$

and five others derived from it; also

$$M(a;\ b_4b_1b_2b_3;\ c_3c_5c_6c_1c_2c_4;\ d_4d_1d_2d_3),$$

and the like five others.

We thus have the comprehensive result that *so long as* a *remains unmoved the* d*'s may be taken in any order whatever, provided that the* b*'s be taken in the same order, and provided that according as the first of the last four places is occupied by* d_1 *or* d_2 *or* d_3 *or* d_4 *the first triad of* c*'s shall be a permutation of* $\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3$ *or* $\mathrm{c}_1\mathrm{c}_4\mathrm{c}_5$ *or* $\mathrm{c}_2\mathrm{c}_4\mathrm{c}_6$ *or* $\mathrm{c}_3\mathrm{c}_5\mathrm{c}_6$ *respectively, and the second triad of* c*'s be when reversed the same permutation of the remaining* c*'s taken in backward order.* (IV.)

For example, if we elect to take the *d*'s in the order $d_4d_2d_1d_3$ then (1) the *b*'s must be written in the order $b_4b_2b_1b_3$, (2) the first triad of c's must be a permutation of $c_3c_5c_6$, these being the *c*'s associated with d_4 in the diagonal of the determinant, (3) the said permutation must be $c_5c_3c_6$, namely, the same permutation as $d_2d_1d_3$ is of $d_1d_2d_3$, (4) the remaining *c*'s in backward order being $c_4c_2c_1$, the same permutation, namely, $c_2c_4c_1$, is taken and then reversed. The final result thus is

$$M(a;\ b_4b_2b_1b_3;\ c_5c_3c_6c_1c_4c_2;\ d_4d_2d_1d_3).$$

10. In the second place we take advantage of the fact that by the performance of certain subtractions on the rows and columns of the determinant there may be obtained new forms of it which still bear all the distinguishing characteristics of *M*, although now the elements themselves have become changed in the course of the transformation. Thus, by subtracting the first row from each of the other rows and thereafter the first column from each of the other columns, we obtain

$$\begin{vmatrix}
d_1+c_1+c_2+c_3 \\ \quad +b_2+b_3+b_4+a & d_1+c_2+c_3+b_2 & d_1+c_1+c_3+b_3 & d_1+c_1+c_2+b_4 \\
d_1+c_2+c_3+b_2 & d_1+b_1+c_2+c_3 \\ \quad +b_2+c_4+c_5+d_2 & d_1+b_1+c_3+c_4 & d_1+b_1+c_2+c_5 \\
d_1+c_1+c_3+b_3 & d_1+b_1+c_3+c_4 & d_1+b_1+c_1+c_3 \\ \quad +b_3+c_4+c_6+d_3 & d_1+b_1+c_1+c_6 \\
d_1+c_1+c_2+b_4 & d_1+b_1+c_2+c_5 & d_1+b_1+c_1+c_6 & d_1+b_1+c_1+c_2 \\ \quad +b_4+c_5+c_6+d_4
\end{vmatrix}$$

which is $$M(d_1;\ b_1c_1c_2c_3;\ b_2b_3b_4c_4c_5c_6;\ ad_2d_3d_4),$$

and where every element, except that in the place 1,1, is different from what it was before. Similarly, by subtracting the second row and second column, there is obtained

$$M(d_2;\ c_1b_2c_4c_5;\ b_1c_2c_3b_3b_4c_6;\ d_1ad_3d_4),$$

where only the element in the place 2,2 is unchanged. Also, by subtracting the third row and third column, we reach

$$M(d_3;\ c_2c_4b_3c_6;\ c_1b_1c_3b_2c_5b_4;\ d_1d_2ad_4),$$

and, by subtracting the fourth row and fourth column,

$$M(d_4;\ c_3c_5c_6b_4;\ c_1c_2b_1c_4b_2b_3;\ d_1d_2d_3a).$$

These amount to saying *that* a *can be interchanged with any* d *provided that the* c*'s associated with that* d *in the original diagonal are interchanged in order with the* b*'s that are also so associated.* (V.)

11. The interchangeability of a with any of the d's is the more curious by reason of the fact that a appears in every one of the sixteen elements of the original determinant and each d only once. It also has considerable practical value, because, while the finding of the co-factor of a from the original determinant would be troublesome, it is an easy matter to do so when the a has changed places with one of the d's, and so occurs in only one of the elements. Thus, from the first of the four forms in § 10, we see that

$$\text{co-factor of } a$$
$$= M(b_1+d_1;\ c_1+c_6,\ c_2+c_5,\ c_3+c_4;\ b_2+d_2,\ b_3+d_3,\ b_4+d_4), \quad \text{(VI.)}$$

and that therefore, from § 7, it consists of 32×8 positive terms. This implies, of course, that a does not occur in a higher power than the first in any term of the final development.

12. The restriction of any other variable to a single diagonal element can be effected in like manner, and, indeed, with less trouble than in the case of a. Taking, for example, the variable c_5, we note first that it occurs in the places

$$2,2 \quad 2,4,$$
$$4,2 \quad 4,4,$$

and we consequently perform the operations

$$\mathrm{row}_2 - \mathrm{row}_4, \quad \mathrm{col}_2 - \mathrm{col}_4,$$

after which it is found only in the place 4,4, and we are able to see that its co-factor is

$$\begin{vmatrix} \left.\begin{matrix} a+b_2+b_3+b_4 \\ +c_1+c_2+c_3+d_1 \end{matrix}\right\} b_4+c_1-b_2-c_3 & a+b_2+b_4+c_2 \\ b_4+c_1-b_2-c_3 & \left.\begin{matrix} b_2+b_4+d_2+d_4 \\ +c_1+c_3+c_4+c_6 \end{matrix}\right\} b_4+c_4-b_2-c_6 \\ a+b_2+b_4+c_2 & b_4+c_4-b_2-c_6 \end{vmatrix} \left.\begin{matrix} \\ \\ a+b_1+b_2+b_4 \\ +c_2+c_4+c_6+d_3 \end{matrix}\right\}.$$

It is thus clear that *a variable cannot occur in a higher power than the first in any term of the final development of M, and that its co-factor is always obtainable in the form of an axisymmetric determinant whose order number is one less than that of M.* (VII.)

13. There is contained in § 11 another fact that is generalisable, the wider result being that *every co-axial minor of a unisignant M is itself a unisignant of the like type.* (VIII.)

In the example there given the minor is of the third order. As another instance let us take the minor just mentioned as being associated with c_5, namely,

$$\begin{vmatrix} \left.\begin{matrix} a+b_1+b_3+b_4 \\ +c_1+c_4+c_5+d_2 \end{matrix}\right\} a+b_1+b_3+c_5 & \\ a+b_1+b_3+c_5 & \left.\begin{matrix} a+b_1+b_2+b_3 \\ +c_3+c_5+c_6+d_4 \end{matrix}\right\} \end{vmatrix}.$$

This is seen to be

$$M(a+b_1+b_3+c_5;\ b_4+c_1+c_4+d_2;\ b_2+c_3+c_6+d_4).$$

14. In connection with the preceding we may note the theorem that, *if all the co-axial minors of a determinant have nothing but positive terms, any positive quantities added to the diagonal elements will not affect the unisignancy* (X.). For, if the quantities so added be x_1, x_2, x_3, ..., the new determinant may be expressed as a sum of terms each of which is

a co-axial minor multiplied by the product of a number of the x's. Examples of determinants that thus produce other unisignants are

$$\begin{vmatrix} \alpha+\beta+\gamma & \alpha-\gamma & \beta-\gamma \\ \alpha-\gamma & \alpha+\gamma+\delta & \gamma-\delta \\ \beta-\gamma & \gamma-\delta & \beta+\gamma+\delta \end{vmatrix} = 9\,(\alpha\beta+\alpha\delta+\beta\delta)\,\gamma,$$

$$\begin{vmatrix} \alpha+\beta+\gamma & \alpha-\gamma & \beta-\gamma \\ \alpha-\gamma & \alpha+\gamma+\delta & \gamma+\delta \\ \beta-\gamma & \gamma+\delta & \beta+\gamma+\delta \end{vmatrix} = 9\alpha\beta\gamma+4\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta,$$

$$\begin{vmatrix} \alpha+\beta+\gamma & \alpha+\gamma & \beta+\gamma \\ \alpha+\gamma & \alpha+\gamma+\delta & \gamma+\delta \\ \beta+\gamma & \gamma+\delta & \beta+\gamma+\delta \end{vmatrix} = \alpha\beta\gamma+4\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta.$$

Thus, from the first of these, by annexing in order x, y, z to the diagonal elements, we have a determinant which equals

$$\begin{aligned} &xyz+xy\,(\beta+\gamma+\delta)+yz\,(\alpha+\beta+\gamma)+zx\,(\alpha+\gamma+\delta) \\ &\quad+x\,\{4\gamma\delta+(\gamma+\delta)\,(\alpha+\beta)+\alpha\beta\} \\ &\quad+y\,\{4\beta\gamma+(\beta+\gamma)\,(\alpha+\delta)+\alpha\delta\} \\ &\quad+z\,\{4\alpha\gamma+(\alpha+\gamma)\,(\beta+\delta)+\beta\delta\} \\ &\quad+9\,(\alpha\beta+\alpha\delta+\beta\delta)\,\gamma, \end{aligned}$$

while from the third we have

$$M\,(\gamma;\ \delta,\ \beta,\ \alpha;\ x,\ y,\ z).$$

15. As the result of § 9 can be used in connection with each of the four separate results of § 10, it is clear that there are obtainable 120 different forms of M, the first place and the last four places being occupiable by any permutation of $a\ d_1\ d_2\ d_3\ d_4$. (IX).

16. The results of §§ 9, 10 may be viewed as propositions regarding the invariance of M, and as such some of them have considerable importance in the discovering and recording of

the final development. Thus the first result in § 9 is equivalent to saying that M *is invariant to the interchange*

$$\begin{pmatrix} b_3 & c_2 & c_4 & d_3 \\ b_4 & c_3 & c_5 & d_4 \end{pmatrix},$$

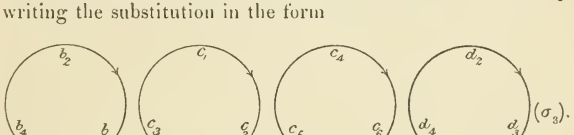
the second result that M *is invariant to the interchange*

$$\begin{pmatrix} b_2 & c_1 & c_5 & d_2 \\ b_3 & c_2 & c_6 & d_3 \end{pmatrix},$$

the third result that M *is invariant to the substitution*

$$\begin{pmatrix} b_2 & b_3 & b_4 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & d_2 & d_3 & d_4 \\ b_3 & b_4 & b_2 & c_2 & c_3 & c_1 & c_6 & c_4 & c_5 & d_3 & d_4 & d_2 \end{pmatrix},$$

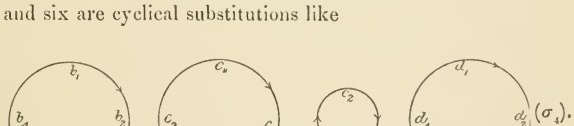
and so on. It is a result like the third that is specially valuable in connection with the obtaining of the terms of the final development, this being due to the cyclical character of the substitution—a character that is best made manifest by writing the substitution in the form



Of the total twenty-three results of § 9 six are interchanges like the first two, eight are cyclical substitutions like the third, three are interchanges like

$$\begin{pmatrix} b_1 & b_2 & c_2 & c_3 & d_1 & d_3 \\ b_2 & b_4 & c_5 & c_4 & d_2 & d_4 \end{pmatrix},$$

and six are cyclical substitutions like



As each of this last group of six cyclical substitutions involves all the variables except a, whereas each of the previous group of eight involves only twelve variables, we are naturally led

to choose for use one of the group of six. Thus, having learned in any way whatever that $ab_4c_6d_1$ is a term of the development, we can, by using a cyclical substitution of the one kind (σ_4), obtain the three other terms $ab_1c_3d_2$, $ab_2c_1d_3$, $ab_3c_4d_4$, whereas, by using one of the other kind (σ_3), we should obtain only two. Of course we might employ both, so that, if $\Sigma ab_4c_6d_1$ were used to denote the one cyclical aggregate and $Sab_4c_6d_1$ the other, we should have $S\Sigma ab_4c_6d_1$ and $\Sigma Sab_4c_6d_1$, each standing for an aggregate of twelve terms, though not necessarily one and the same aggregate.

17. It is almost certain that the best mode of seeking the development of M is that based on the fact that the first variable and the last four are quite similarly involved in the constitution of the function, and therefore ought to be viewed as belonging to one and the same set. Adopting this basis we consequently try to obtain all the terms that are independent of a, d_1, d_2, d_3, d_4; in the second place the terms that involve only one of the set; in the third place the terms that involve only two of the set; and so on. With only the four d's to be considered there would have been no difficulty in following out this procedure, as in the original determinant the d's are confined to the diagonal element, each d being a term of only one element. As matters stand, however, it is necessary to use in addition one of the forms of the determinant obtained in § 10 and also to profit by the cyclical substitution σ_4.

The aggregate of the first desired collection of terms is evidently

$$M(0;\ b_1b_2b_3b_4;\ c_1c_2c_3c_4c_5c_6;\ 0000),$$

the development of which we shall consider later. The aggregate of the terms containing a, but free from d_1, d_2, d_3, d_4, is seen from § 11 to be

$$aM(b_1;\ c_1+c_6,\ c_2+c_5,\ c_3+c_4;\ b_2,\ b_3,\ b_4),$$

the aggregate of the terms containing d_1, but free from a, d_2, d_3, d_4, we know from the original form of M to be

$$d_1M(b_1;\ b_2+c_6,\ b_3+c_5,\ b_4+c_4;\ c_1,\ c_2,\ c_3),$$

and similar aggregates for d_2, d_3, d_4 can be got from the same source or by means of the cyclial substitution. Again, the

aggregate of terms involving both a and d_2, but no others of the set, is seen from § 10 to be

$$ad_2M(b_1+c_1+c_6,\ b_3+c_3+c_4,\ b_4+c_2+c_5),$$

from which by cyclical substitution we obtain the corresponding expressions connected with ad_3, ad_4, ad_1. Similarly, from the original form of M, the analogous expressions

$$d_1d_2M(b_1+b_2+c_6,\ b_4+c_2+c_4,\ b_3+c_3+c_5),$$

$$d_1d_3M(b_1+b_3+c_6,\ b_4+c_1+c_4,\ b_2+c_3+c_6)$$

are found, the former then giving by cyclical substitution three others, and the latter one other. The aggregates involving three and the aggregates involving four variables of the set are similarly and readily obtainable; and the collected result as thus far expressible is consequently

$$\begin{aligned}
M(0;\ b_1b_2b_3b_4;\ c_1c_2c_3c_4c_5c_6;\ 0000)\\
&+\quad aM(b_1;\ c_1+c_6,\ c_2+c_5,\ c_3+c_4;\ b_2,\ b_3,\ b_4))\\
&+\Sigma d_1M(b_1;\ b_2+c_6,\ b_3+c_5,\ b_4+c_4;\ c_1,\ c_2,\ c_3)\Big)\\
&+\ \Sigma ad_1M(b_2+c_6+c_1,\ b_3+c_5+c_2,\ b_4+c_4+c_3)\Big)\\
&+\Sigma d_1d_2M(b_1+b_2+c_6,\ b_4+c_2+c_4,\ b_3+c_3+c_5)\Big\}\\
&+\Sigma d_1d_3M(b_1+b_3+c_5,\ b_4+c_1+c_4,\ b_2+c_3+c_6)\Big)\\
&+\ \Sigma ad_1d_2(b_3+b_4+c_2+c_3+c_4+c_5)\Big)\\
&+\ \Sigma ad_1d_3(b_2+b_4+c_1+c_3+c_4+c_6)\Big\}\\
&+\Sigma d_1d_2d_3(b_1+b_2+b_3+c_3+c_5+c_6)\Big)\\
&+\Sigma ad_1d_2d_3+d_1d_2d_3d_4. \qquad \text{(XI.)}
\end{aligned}$$

Since M's of the third order are already known to contain only positive terms, it is seen that every term obtainable from the last nine lines of this expression bears the sign +. Further, it is readily found that the number of terms containing only one variable of the set $ad_1d_2d_3d_4$ is 108×5, the number containing two variables is 27×10, the number containing three variables is 6×10, and the number containing four variables is 5: in all 875.

18. As for the remaining line of (XI.), namely, the determinant

$$\begin{vmatrix} b_2+b_3+b_4+c_1+c_2+c_3 & b_3+b_4+c_1 & b_2+b_4+c_2 & b_2+b_3+c_3 \\ b_3+b_4+c_1 & b_1+b_3+b_4+c_1+c_4+c_5 & b_1+b_4+c_4 & b_1+b_3+c_5 \\ b_2+b_4+c_1 & b_1+b_4+c_4 & b_1+b_2+b_4+c_2+c_4+c_6 & b_1+b_2+c_6 \\ b_2+b_3+c_3 & b_1+b_3+c_5 & b_1+b_2+c_6 & b_1+b_2+b_3+c_3+c_5+c_6 \end{vmatrix},$$

we first find the aggregate of the terms which are independent of the b's, namely,

$$4\overset{\circ}{\Sigma}c_1c_2c_3(c_4+c_5+c_6)$$

or

$$4\{c_1c_6(c_2+c_5)(c_3+c_4)+c_2c_5(c_3+c_4)(c_1+c_6)+c_3c_4(c_1+c_6)(c_2+c_5)\}.$$

We then transform the determinant in the manner specified in §12 so as to have b_1 in the place 4,4 only, the result of this being that the

co-factor of b_1 equals

$$\begin{vmatrix} b_2+b_3+b_4+c_1+c_2+c_3 & b_4-b_2+c_1-c_3 & b_4-b_3+c_2-c_3 \\ b_4-b_2+c_1-c_3 & b_2+b_4+c_1+c_3+c_4+c_6 & b_4+c_4+c_3 \\ b_4-b_3+c_2-c_3 & b_4+c_4+c_3 & b_3+b_4+c_2+c_3+c_4+c_5 \end{vmatrix},$$

whence, in passing, the c co-factor of b_1 is (§14) readily obtained, namely,

$$\begin{aligned} c_5c_6(c_1+c_2+c_3)+c_5\{4c_3c_1+(c_3+c_1)(c_2+c_4)+c_2c_4\} \\ +c_6\{4c_3c_2+(c_3+c_2)(c_1+c_4)+c_1c_4\} \\ +9c_1c_2c_3+4c_1c_2c_4+c_1c_3c_4+c_2c_3c_4. \end{aligned}$$

Next, treating the whole co-factor of b_1 so as to isolate b_2 and b_3, namely, by performing the operations

$$\text{row}_2+\text{row}_1, \quad \text{row}_1+\text{row}_3,$$

$$\text{col}_2+\text{col}_1, \quad \text{col}_1+\text{col}_3,$$

we find the said co-factor to be

$$\begin{vmatrix} b_2+4b_4+c_1+4c_2+c_4+c_5 & 6b_4+2c_1+4c_2+2c_4+c_5 & 2b_4+2c_2+c_4+c_5 \\ 6b_4+2c_1+4c_2+2c_4+c_5 & 9b_4+4(c_1+c_2+c_4)+c_3+c_5+c_6 & 3b_4+2c_2+c_3+2c_4+c_5 \\ 2b_4+2c_2+c_4+c_5 & 3b_4+2c_2+c_3+2c_4+c_5 & b_3+b_4+c_2+c_3+c_4+c_5 \end{vmatrix},$$

and thence see that the

$$c \text{ co-factor of } b_1b_2 = \begin{vmatrix} 4(c_1+c_2+c_4)+c_3+c_5+c_6 & 2c_2+c_3+2c_4+c_5 \\ 2c_2+c_3+2c_4+c_5 & c_2+c_3+c_4+c_5 \end{vmatrix}$$

$$= \begin{vmatrix} 4c_1+2c_2+2c_4+c_6 & c_2+c_4 \\ 2c_2+c_3+2c_4+c_5 & c_2+c_3+c_4+c_5 \end{vmatrix}$$

$$= \begin{vmatrix} 4c_1+c_2+c_4+c_6 & c_2+c_4 \\ c_2+c_4 & c_2+c_3+c_4+c_5 \end{vmatrix}$$

$$= M(c_2+c_4,\ 4c_1+c_6,\ c_3+c_5),$$

c co-factor of $b_1b_3 = M(c_1+c_4,\ 4c_2+c_5,\ c_3+c_6)$,

and the whole co-factor of $b_1b_2b_3$

$$= 9b_4+4(c_1+c_2+c_4)+c_3+c_5+c_6.$$

The result desired is thus

$$M(0;\ b_1b_2b_3b_4;\ c_1c_2c_3c_4c_5c_6;\ 0000) \qquad \text{(XII.)}$$

$$= 9b_1b_2b_3b_4 + \overset{\circ}{\Sigma} b_1b_2b_3(4c_1+4c_2+4c_4+c_3+c_5+c_6)$$

$$+ \overset{\circ}{\Sigma} b_1b_2\{(c_3+c_4)(4c_1+c_6)+(4c_1+c_6)(c_3+c_5)+(c_3+c_5)(c_2+c_4)\}$$

$$+ \overset{\circ}{\Sigma} b_1b_3\{(c_1+c_4)(4c_2+c_5)+(4c_2+c_5)(c_3+c_6)+(c_3+c_6)(c_1+c_4)\}$$

$$+ \overset{\circ}{\Sigma} b_1 \begin{cases} (c_1c_6+c_2c_5)(c_3+c_4)+(c_2c_5+c_3c_4)(c_1+c_6)+(c_3c_4+c_1c_6)(c_2+c_5) \\ \quad +c_2c_6(4c_3+c_4)+c_3c_5(4c_1+c_6)+c_4c_1(4c_2+c_5) \\ \quad +9c_1c_2c_3 \end{cases}$$

$$+ 4\{c_1c_6(c_2+c_5)(c_3+c_4)+c_2c_5(c_3+c_4)(c_1+c_6)+c_3c_4(c_1+c_6)(c_2+c_5)\},$$

the terms being evidently all positive and the number of them

$$9 + 15\times4 + 24\times6 + 36\times4 + 12\times4, \quad i.e.,\ 405.$$

This agrees with the number obtainable on putting each of the variables equal to 1 in the determinant form, namely,

$$\begin{vmatrix} 6&3&3&3 \\ 3&6&3&3 \\ 3&3&6&3 \\ 3&3&3&6 \end{vmatrix}, \quad i.e.,\ 3^4.5 \qquad \begin{vmatrix} 1&1&1&1 \\ 1&2&1&1 \\ 1&1&2&1 \\ 1&1&1&2 \end{vmatrix}, \quad i.e.,\ 405.$$

An examination of the form of the development in (XII.) brings out the fact that of 210 different sets of four variables that may be chosen from the given ten there are only 25 sets which are excluded from forming a term. These are 3 sets consisting of c's only

$$c_1c_6c_2c_5, \quad c_2c_5c_3c_4, \quad c_3c_4c_1c_6;$$

4 sets consisting of one b and three c's

$$b_1c_4c_5c_6, \quad b_2c_2c_3c_6, \quad b_3c_1c_3c_5, \quad b_4c_1c_2c_4;$$

and 18 sets consisting of two b's and two c's

$$b_1b_2c_1c_6, \ b_1b_2c_2c_4, \ b_1b_2c_3c_5, \ \ldots,$$
$$b_1b_3c_1c_4, \ b_1b_3c_2c_5, \ b_1b_3c_3c_6, \ \ldots,$$

the others of the 18 being obtained by the circular substitution.

19. The total number of terms of the M of the fourth order is thus seen to be $875 + 405$; that is 1280, another result which is verifiable in similar fashion, namely, by evaluating

$$M(1\,;\ 1111\,;\ 111111\,;\ 1111),$$

which gives the $4^4.5$. *In an M of the n^{th} order the number of terms is* $2^{n(n+2)}.(n+1)$. (XIII.)

Cape Town, S.A.,
December 12th, 1910.

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